A practical method for the valuation of a variety of hybrid products

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Contents

I. Hybrid products for retail and wholesale investors

Both retail and wholesale investors show demand for capital guaranteed notes with participation in the

best of or a ***ranked weighting of***

performances of different underlyings such as:-

- **(averaged) equity**
- **inflation indices**
- **FX rates**
- **commodity returns**
- **total returns on fixed income products**

II. Minimal process assumption valuation

One of the greatest challenges in hybrid modelling is to accomodate the differing specific needs in the respective underlying markets.

In particular, for a product involving several different markets, one might have to combine:-

- local volatility (e.g. equity)
- stochastic volatility (e.g. equity and/or FX)
- mean reversion (e.g. commodity and/or fixed income)
- change-of-measure considerations (most important when fixed income components are involved)

In the face of the sheer complexity of this task, it helps to go back to the basics of derivatives trading and hedging:-

- consistent pricing of relevant hedge instruments (e.g. forwards and plain vanilla options)
- selection of justifiable modelling assumptions (e.g. co-dependence structure aka correlation)
- considerations regarding *enforceable* arbitrage

Calibration to plain vanilla options

Assuming that we can justify a risk-neutral par forward contract strike, and the specification of a Black implied volatility surface in the underlying asset classes as a function of maturity, we know the risk-neutral distribution of each of the underlyings [BL78] in their own natural measure.

We hereby define the natural measure of any variable the measure that makes the variable itself a martingale.

For instance, for an equity dependent payoff we would have, in the measure induced by the choice of numéraire being the zero coupon bond $P_T(t)$ maturing on the option's payment date T,

$$\begin{aligned} \text{Call} &= P_T(0) \cdot \mathsf{E}_T^{\mathcal{M}(P_T)}\big[(F_T(T) - K)_+\big] \\ &= P_T(0) \cdot [F_T(0) \cdot \Phi(d_1) - K \cdot \Phi(d_2)] \end{aligned} \quad (1)$$

with $F_T(t)$ being the par strike of a forward contract maturing at T as seen at time t (which implies $F_T(T) = S(T)$) and

$$d_1 \quad := \quad \frac{\ln F_T(0) - \ln K + \frac{1}{2}\hat{\sigma}^2(K,T)T}{\hat{\sigma}(K,T)\sqrt{T}} \tag{2}$$

$$d_2 \quad := \quad d_1 - \hat{\sigma}(K,T)\sqrt{T}\ . \tag{3}$$

From this we have directly

$$\mathsf{Prob}\left\{F_T(T) > K\right\} \quad = \quad -\partial_K \mathsf{E}_T^{\mathcal{M}(P_T)}\left[\left(F_T(T) - K\right)_+\right] \tag{4}$$

$$\mathsf{Prob}\left\{F_T(T) < K\right\} \quad = \quad 1 - \mathsf{Prob}\left\{F_T(T) > K\right\} \tag{5}$$

$$\mathsf{Prob}\left\{F_T(T) < K\right\} \quad = \quad \Phi\left(-d_2\right) + F_T(0)\cdot\sqrt{T}\cdot\varphi\left(d_1\right)\cdot\partial_K\hat{\sigma}(K,T)\ . \tag{6}$$

We thus gain access to the risk-neutral quantile function

$$q_T(K) \;:=\; \text{Prob}\{F_T(T) < K\} \tag{7}$$

which is nothing other than the cumulative probability function.

In other words, the quantile function maps the quantile level to the cumulative probability.

The inverse quantile function $q_t^{-1}(p)$, in turn, maps the cumulative probability to the quantile level of the underlying.

The inverse quantile function

$$q_t^{-1}(p)$$

enables us to map the quantile of any distribution to the associated level in the respective underlying consistent with its specific risk-neutral distribution.

Drawing $S(t)$ as $q_t^{-1}(u)$ with $u \sim \mathcal{U}(0,1)$ ensures that we

reprice all plain vanilla options for all strikes by construction!

Given a payoff formula

$$\pi(S(T))$$

that only depends on a single fixing, and the risk-neutral distribution of $F_T(T) = S(T)$ in its own natural measure whose inducing numéraire be denoted by N, the value of the contingency claim that pays $\pi(S(T))$ at T is

$$V = N(0) \cdot \mathsf{E}_T^{\mathcal{M}(N)}[\pi(S(T))/\,N(T)] \tag{8}$$

by virtue of the fundamental theorem of asset pricing [HP81]. In terms of calculus, this means

$$V = N(0) \cdot \int \frac{\pi(S)}{N(T)} q_T'(S)\,\mathrm{d}S \tag{9}$$

$$V = N(0) \cdot \int \frac{\pi(q_T^{-1}(u))}{N(T)}\,\mathrm{d}u\;. \tag{10}$$

Clearly, $q_T'(S) = \partial_S q_T(S)$ is simply the risk-neutral density of S for maturity T.

The spatial copula

When multiple underlyings are involved, as would be the case in a hybrid derivative, risk-neutral marginal densities have to be connected to form a multivariate distribution. A general framework for the specification of the co-dependence of two marginal distributions is that of a *copula*.

- In a model that is based on the initial specification of underlying stochastic processes, and their co-dependence, the interdependence structure of marginal distributions of multiple underlyings on any given observation time is a result of the process assumptions.
- A process-created co-dependence structure is also consistent with the framework of a copula, only that we have no external direct control on this co-dependence structure.
- We can only modify the process-specific correlation coefficients, and take the effect this has on the co-dependence of marginal distributions as an output of the model.
- In contrast to that, when we specify the marginal distributions directly, we still have full control over the co-dependence structure we favour.
- Since the fine structure of co-dependence between underlying assets from different market segments (such as CPI and equity) is to a large extent unknown, any choice of correlation, whether given by underlying instantaneous processes or by the direct specification of a copula connecting marginal distributions, is, to some extent, arbitrary, and thus part of the modelling assumptions that we are free to make.

For reasons of tractability and simplicity we choose the Gaussian copula for inter-asset co-dependence modelling.

In the special but not uncommon case that we use a Black implied volatility profile without skew or smile, the connection of the corresponding lognormal distributions via a Gaussian copula is mathematically exactly what one obtains from correlated geometric Brownian motions, which is a sensible benchmark feature to have for the modelling of co-dependence.

Both linear and log-linear correlation coefficients change not only if the connecting copula is altered (structurally or parametrically), but also when the marginal distributions change independently (change in implied volatility skew or smile).

➜ It is generally advisable to use a *rank correlation* coefficient as a measure of co-dependence instead of conventional correlation figures.

One such rank correlation measure is known as *Spearman's rho* and is nothing other than the linear correlation computed from the marginal probabilities, i.e. $u_i = q_T^{(i)}(S_i)$.

Spearman's rho $\rho_{u_i u_j}$ can be converted from and to the Gaussian correlation coefficient $\varrho_{z_i z_j}$ directly:-

$$\rho_{u_i u_j} = \frac{6}{\pi} \cdot \arcsin\left(\frac{1}{2} \cdot \varrho_{z_i z_j}\right) \quad \text{and} \quad \varrho_{z_i z_j} = 2 \cdot \sin\left(\frac{\pi}{6} \cdot \rho_{u_i u_j}\right) . \tag{11}$$

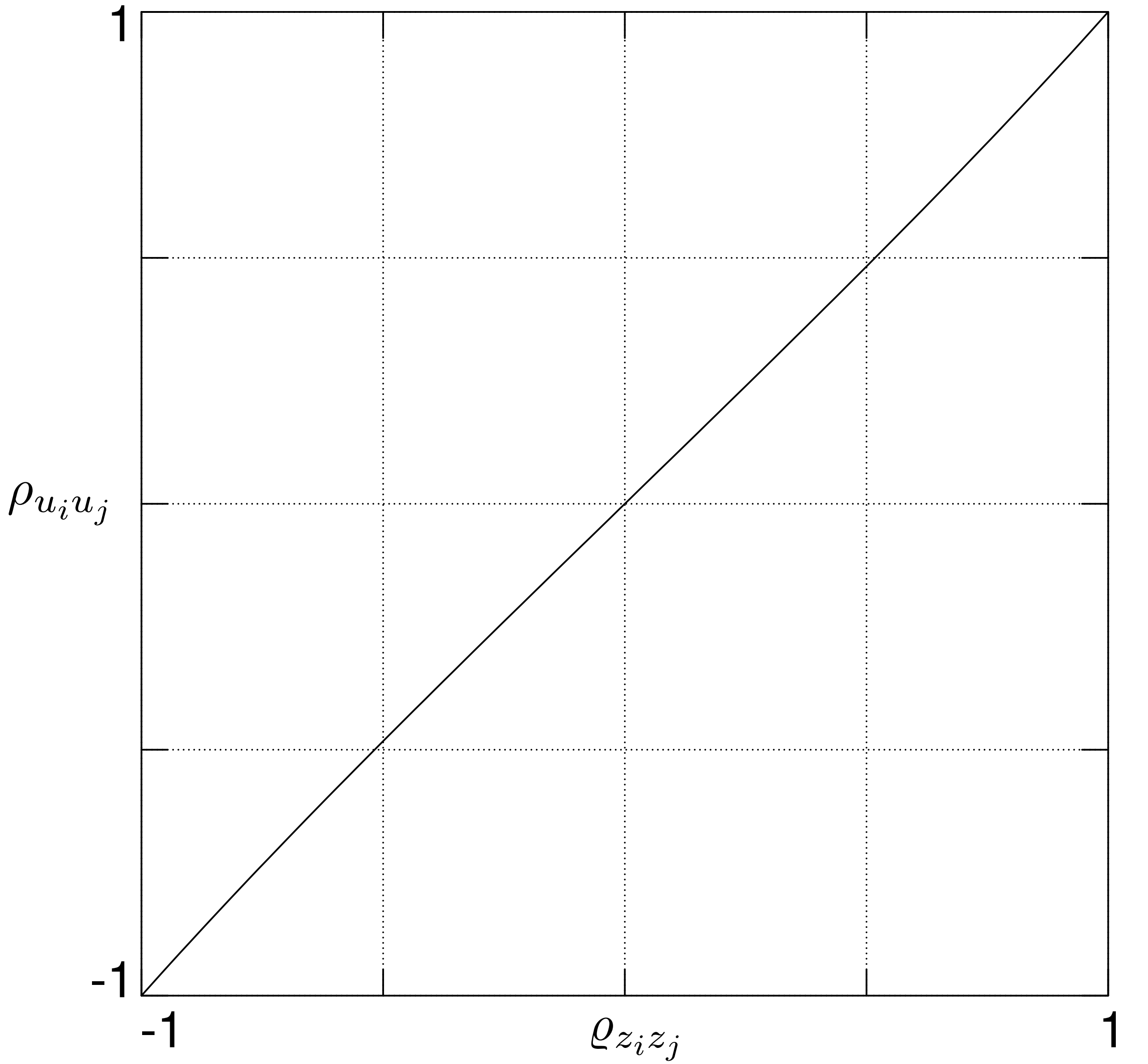


Figure II.1. Spearman's rho of correlated Gaussian variates. Not quite a straight line, but nearly.

The temporal copula

The quantiles $q_t(S(t))$ for different time horizons t are codependent standard uniform variates.

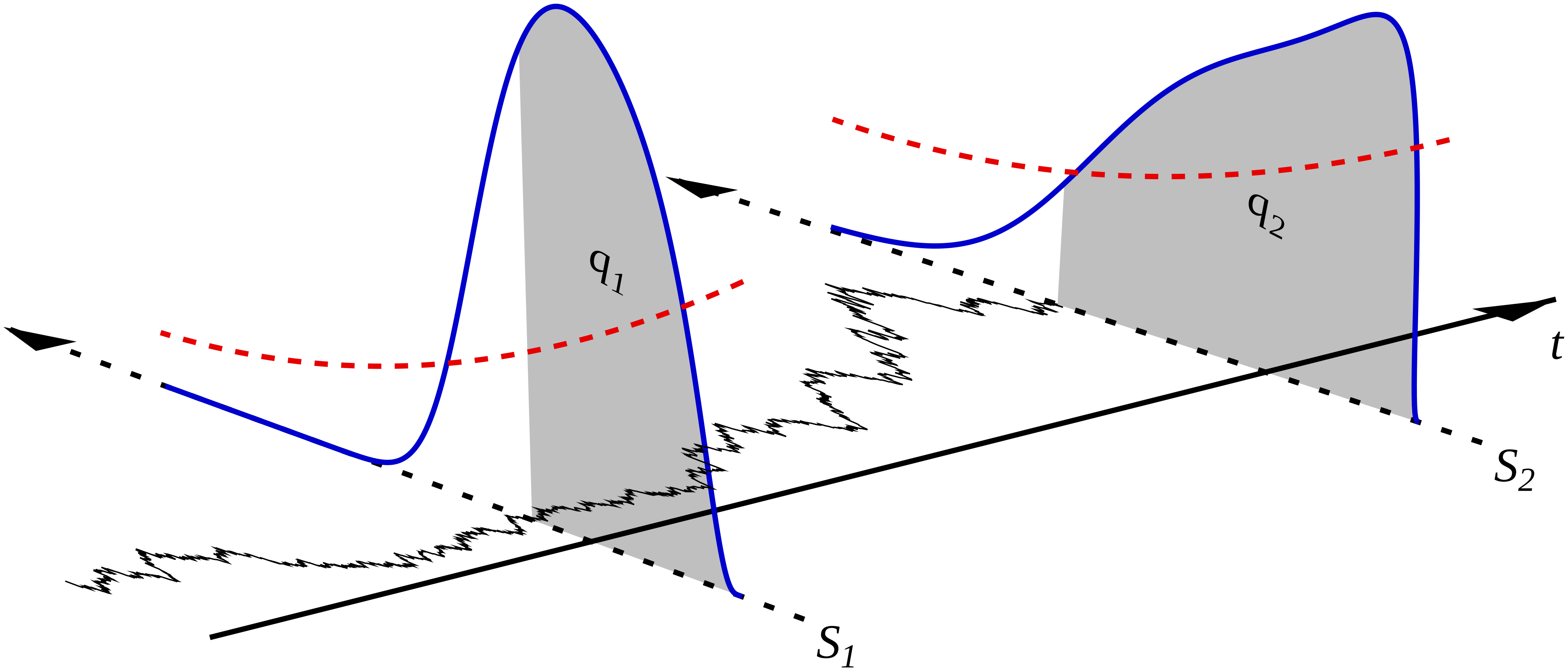


Figure II.2. Blue line: risk-neutral density. Red dashed line: implied volatility.

Define

$$z(t) \quad := \quad \Phi^{-1}(q_t(S(t))) \tag{12}$$

For zero skew and smile, the variates z_t are normally distributed with correlation structure

$$\hat{\varrho}(t_i, t_j) \quad = \quad \frac{\hat{\sigma}(t_i)}{\hat{\sigma}(t_j)}\sqrt{\frac{t_i}{t_j}} \qquad \text{for any} \quad t_i < t_j \ . \tag{13}$$

As a lowest order approximation for the codependence between the different z_{t_i}, we can use a Gaussian copula!

However,

how do we draw $S(t) \ = \ q_t^{-1}(\Phi(z_t))$ efficiently?

For each observed time horizon t_i:-

1. Solve for a lower limit quantile level $S_l(t_i)$ using (7) such that

$$q_{t_i}(S_l(t_i)) \; = \; \varepsilon$$

with ε being a fraction of the smallest uniform number possibly to be drawn from the uniform number generator, i.e. $\varepsilon \approx 2^{-33}$.

2. Solve

$$q_{t_i}(S_u(t_i)) \; = \; 1 - \varepsilon$$

for $S_u(t_i)$.

3. Using the above endpoints, compute two vectors S[·] and z[·] of associated interpolation points such that z[k] = $\Phi^{-1}(q_{t_i}($S[k]$))$.

4. Instantiate a monotonicity preserving interpolation [Hym83, Kva00] object that interpolates $S = S(z)$, thus numerically inverting the normal-variate-equivalent-quantile map.

The required interpolation table can be constructed very efficiently:

```
//
// Equation (12).
//
double inverseCumulativeNormalOfQuantile(double spot) const;

//
// Given two vectors S[·] and z[·], each initialised with two elements such that
// S[0]=S_l, S[1]=S_u, z[0]=Φ^{-1}(q_t(S_l)), and z[1]=Φ^{-1}(q_t(S_u)), recursively insert points
// until linear interpolation is sufficiently accurate. Invoke with location=1.
//
void insertAtLeastOneMorePoint( std::vector <double> & S, std::vector <double> & z,
                                unsigned long location, double accuracy ) {
  const double z_mid = 0.5*(z[location-1]+z[location]);
  S.insert( S.begin()+location, 0.5*(S[location-1]+S[location]) );
  z.insert( z.begin()+location, inverseCumulativeNormalOfQuantile(S[location]) );
  if ( fabs(z_mid-z[location]) > accuracy ) {  // Recursion if necessary.
    insertAtLeastOneMorePoint( S, z, location+1, accuracy );
    insertAtLeastOneMorePoint( S, z, location, accuracy );
  }
}
```

III. Theoretical arbitrage considerations

There is only one catch:

This method does not guarantee risk-neutral drift conditions!

In particular, for skewed implied volatility surfaces, it does not preserve

$$\mathsf{E}_0\left[\frac{S(t_i)}{S(t_j)}\right] = \frac{F(t_i)}{F(t_j)}, \tag{14}$$

or even

$$\mathsf{E}_t\left[\frac{\mathrm{d}S(t)}{S(t)\mathrm{d}t}\right] = \mu(t) \qquad \text{with} \qquad \mu(t) := \frac{\dot{F}(t)}{F(t)} \tag{15}$$

This can be important for cliquets and strongly path dependent options!

However, $\mathsf{E}_0\left[\frac{\mathrm{d}S(t)}{S(t)\mathrm{d}t}\right] = \mu(t)$ can be salvaged (semi-)analytically!

Define

$$y := \int_0^t \omega(s)\,\mathrm{d}W_s \tag{16}$$

$$v := \int_0^t \omega^2(s)\,\mathrm{d}s \tag{17}$$

$$z := \frac{y}{\sqrt{v}} \tag{18}$$

for some arbitrary function $\omega(t)$.

The new function $\omega(t)$ is a time-changer that effectively serves for a consistent[1] rescaling of the auto-correlation structure of $z(t)$.

Changing the time scale of the process y driving $S(t)$ is all we can do if we want to retain the property that $S(t)$ is Markovian in $y(t)$.

[1] i.e. avoiding non-positive-semidefiniteness

The process z is normally distributed for all $t > 0$ and satisfies

$$\mathrm{d}z = \frac{\omega}{\sqrt{v}}\mathrm{d}W - \frac{1}{2}z\frac{\omega^2}{v}\mathrm{d}t \; . \tag{19}$$

Applying Itô's lemma to both sides of

$$\Phi(z) = q(S,t) \; ,$$

we can derive the stochastic differential equation for $S(t)$:

$$\mathrm{d}\boldsymbol{S} = (\boldsymbol{\zeta} + \boldsymbol{\nu}\boldsymbol{\eta})\,\boldsymbol{S}\,\mathrm{d}\boldsymbol{t} + \sqrt{\boldsymbol{\nu}} \cdot \frac{\boldsymbol{\varphi}}{\boldsymbol{q'}}\,\mathrm{d}\boldsymbol{W} \tag{20}$$

with

$$\begin{aligned} \nu &:= \omega^2/v & \zeta &:= -\frac{\dot{q}}{Sq'} \\ z &= \Phi^{-1}(q) & & \\ \varphi &:= \Phi'(z) & \eta &:= -\frac{\varphi}{Sq'}\left(z + \frac{1}{2}\frac{\varphi q''}{q'^2}\right) \quad . \end{aligned} \tag{21}$$

Alas, for arbitrary (but arbitrage-free) implied volatility surfaces $\hat{\sigma}(K,T)$,

$$\zeta(S,t) + \nu(t)\eta(S,t) \quad \neq \quad \mu(t) \tag{22}$$

for any choice of $\nu(t)$. However, if we compute

$$\bar{\zeta}(t) \quad := \quad \mathsf{E}_0[\zeta(S,t)] \qquad \text{and} \qquad \bar{\eta}(t) \quad := \quad \mathsf{E}_0[\eta(S,t)] \ , \tag{23}$$

we can restore at least

$$\mathsf{E}_0[\mathrm{d}S(t)/S(t)] = \mu(t)\,\mathrm{d}t \tag{24}$$

by setting

$$\nu(t) \quad := \quad \left[\mu(t) - \bar{\zeta}(t)\right] / \,\bar{\eta}(t) \tag{25}$$

and solving the ordinary differential equation

$$\mathrm{d}\ln v \quad = \quad \nu\,\mathrm{d}t \tag{26}$$

numerically for the time-changing variance $v(t)$.

Note that the function $\nu(t)$ becomes singular for $t \to 0$ since it was defined in (21) as

$$\nu(t) \;=\; \frac{\omega^2(t)}{\int_0^t \omega^2(s)\,\mathrm{d}s}\;. \tag{27}$$

This singularity is irrelevant, though, since the value $v(t_1)$ (with t_1 being the first discrete observation time) can be chosen arbitrarily (as long as it is positive).

Ultimately, the model's implementation needs only draw variates for a discrete set of points in time. This can be done solely based on the values of v on the observation times since the auto-correlation structure of the model is fully determined by

$$\hat{\varrho}_{z(t_i)\,z(t_j)} \;=\; \sqrt{\frac{v(t_i)}{v(t_j)}} \qquad \text{for any} \quad t_i < t_j\;. \tag{28}$$

IV. Enforceable arbitrage considerations

For many underlyings, the most important practical no-arbitrage condition is, the one given in equation (14), i.e.

$$\mathsf{E}_0^{\mathcal{M}(\beta)}\left[\frac{S(t_j)}{S(t_i)}\right] = \frac{F_{t_j}(0)}{F_{t_i}(0)} ,$$

where $\beta(t)$ is the continuously rolled up money market account.

Failing this condition significantly would expose the model to the following relative value trading opportunity (assuming deterministic interest rates).

FORWARD PERFORMANCE RELATIVE VALUE STRATEGY:

- AT $t = 0$, INVEST $P_{T_1}(0)$ CURRENCY UNITS IN ZERO COUPON BONDS FOR MATURITY AT T_1.
- AT T_1, THE MATURING SINGLE CURRENCY UNIT IS USED TO BUY $1/S(T_1)$ OF THE UNDERLYING.
- ASSUMING NO CONVENIENCE YIELD ON THE INVESTMENT, AT T_2, THE INVESTMENT IS WORTH EXACTLY $S(T_2)/\,S(T_1)$.

This strategy must not, in expectation, give rise to any profit within the model whence the value of a contract paying $S(T_2)/\,S(T_1)$ at T_2 must be $P_{T_1}(0)$.

A refinement of the argument allowing for deterministic convenience yields leads to the value having to satisfy $P_{T_2}(0) \cdot F_{T_2}(0)/\,F_{T_1}(0)$ which means that the model must satisfy (14) to be protected against the explained relative value trade.

This can be seen as follows:

A par forward contract strike $F_T(0)$ for maturity at T as seen at inception $t = 0$ is given by the current underlying price divided by the product of a T-dated zero coupon bond and a deterministic growth factor due to the assumed deterministic (continuously reinvested) yield on the underlying.

This means that any number N of units of the underlying held at T_1, will, due to its continuous proportional (reinvested) yield, grow from T_1 to T_2 to a total of $\left(\frac{F_{T_1}(0)P_{T_1}(0)}{F_{T_2}(0)P_{T_2}(0)}\right) \cdot N$ units.

Thus, in order to hold at T_2 exactly $1/S(T_1)$ units of the underlying of net value $S(T_2)/S(T_1)$, one needs to buy, at inception, exactly $\left(\frac{F_{T_2}(0)P_{T_2}(0)}{F_{T_1}(0)P_{T_1}(0)}\right)$ zero coupon bonds maturing at T_1 whose proceeds at T_1 are to be invested into the asset under consideration.

Equation (14) can be satisfied by directly adjusting the correlation numbers $\hat{\varrho}(t_i, t_j)$ that are used in the time copula.

This can be done with the aid of a simple one-dimensional root finding procedure. The calculation of the expectation $\mathsf{E}_0^{\mathcal{M}(\beta)}\left[\frac{S(t_j)}{S(t_i)}\right]$ required in each iteration can be simplified by the aid of the following representation:

$$\mathsf{E}_0^{\mathcal{M}(\beta)}\left[\frac{S(t_j)}{S(t_i)}\right] = \iint \frac{q_{t_j}^{-1}(\Phi(\hat{\rho}_{ij}x + \hat{\rho}'_{ij}y))}{q_{t_i}^{-1}(\Phi(x))} \cdot \varphi(x)\varphi(y)\,\mathrm{d}x\,\mathrm{d}y \tag{29}$$

$$= \int \left[\int \frac{\varphi(x)}{q_{t_i}^{-1}(\Phi(-x))} \cdot q_{t_j}^{-1}(\Phi(\hat{\rho}_{ij} \cdot (\varkappa_{ij}y - x)))\,\mathrm{d}x\right] \varphi(y)\,\mathrm{d}y \tag{30}$$

$$= \int \theta\left(\varkappa_{ij} \cdot y\right) \cdot \varphi(y)\,\mathrm{d}y \tag{31}$$

wherein we have defined

$$\begin{aligned} \hat\rho_{ij} &:= \hat\rho(t_i,t_j) & \hat\rho'_{ij} &:= \sqrt{1-\hat\rho_{ij}^2} \\ \varkappa_{ij} &:= \frac{\hat\rho'_{ij}}{\hat\rho_{ij}} & f(x) &:= \frac{\varphi(x)}{q_{t_i}^{-1}(\Phi(-x))} \\ g(\xi-x) &:= q_{t_j}^{-1}(\Phi(\hat\rho_{ij}\cdot(\xi-x))) & \theta(\xi) &:= \int f(x)g(\xi-x)\,\mathrm{d}x\ . \end{aligned} \tag{32}$$

The function $\theta(\xi)$ is a convolution expression that can be computed in a single step for many values in ξ by the aid of fast Fourier transformations. The calculation is then reduced to a one-dimensional Gauss-Hermite quadrature of the function $\theta(\varkappa_{ij}y)$ in equation (31).

Alternatively, solving for the appropriate correlation number $\hat\varrho(t_i,t_j)$ can also be done using a bivariate Gauss-Hermite quadrature for the calculation of $\mathrm{E}_0^{\mathcal{M}(\beta)}\left[\frac{S(t_j)}{S(t_i)}\right]$ in each iteration of the root-finding procedure.

In practice, bivariate Gauss-Hermite quadrature can be made so robust and fast [Jäc05] that the more complicated procedure above is not necessary.

V. Interest rate convexity effects

In general, especially when the hybrid product contains interest rate components, the value of the numéraire $N(T)$ is not independent of $S(T)$.

This means,

$$\mathsf{E}_T^{\mathcal{M}(N)}[1/\,N(T)] \quad \neq \quad P_T(0)/\,N(0)$$

which makes it harder to reduce the evaluation of (10) to the quadrature problem

$$\int \pi(q_T^{-1}(u))\,\mathrm{d}u\ .$$

In this case, we may need to resort to approximate representations of $1/\,N(T)$ as a functional form of $S(T)$ such as the commonly used linear swap rate, constant yield, or similar approximations.

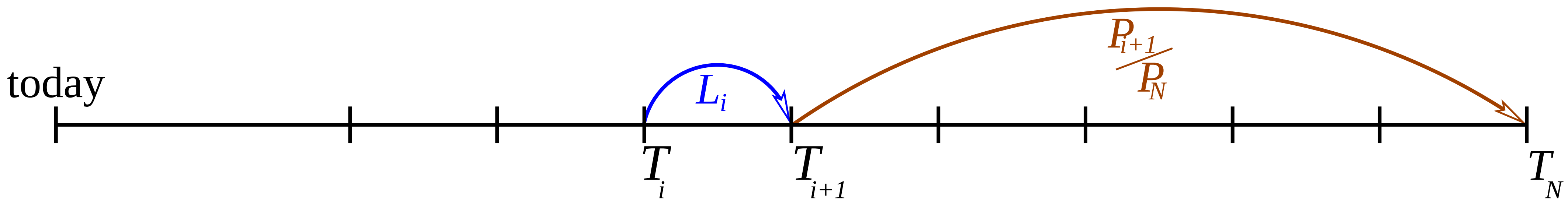
There are, however, product variations that make the use of measure change techniques unnecessary!

Example:

A client is interested in the total return on a swap as part of a basket.

➜ As a proxy, one can use the sum of future Libor fixings, each of which is rolled up at a zero coupon rate (determined at the time of its fixing) from its natural payment time to the performance basket evaluation time:

$$R \;:=\; \sum_{i=1}^{N-1} L_i(T_i) \cdot \frac{P_{T_{i+1}}(T_i)}{P_{T_N}(T_i)} \tag{33}$$



Valuation of all Libor constituents in the terminal measure $\mathcal{M}(P_{T_N})$ is easy since the risk-neutral distribution of $L_i(T_i) \cdot \frac{P_{T_{i+1}}(T_i)}{P_{T_N}(T_i)}$ in the measure $\mathcal{M}(P_{T_N})$ is the same as the risk-neutral distribution of $L_i(T_i) \cdot \frac{P_{T_{i+1}}(0)}{P_{T_N}(0)}$ in the measure $\mathcal{M}(P_{T_{i+1}})$:

$$\left(L_i(T_i) \cdot \frac{P_{T_{i+1}}(T_i)}{P_{T_N}(T_i)} \middle| \mathcal{F}_{T_i} \right)^{\mathcal{M}(P_{T_N})} \hat{=} \left(L_i(T_i) \cdot \frac{P_{i+1}(0)}{P_N(0)} \middle| \mathcal{F}_{T_i} \right)^{\mathcal{M}(P_{T_{i+1}})} \tag{34}$$

- ➜ **The measure $\mathcal{M}(P_{T_{i+1}})$ is the natural measure of L_i !**
- ➜ **The quantile map for the Libor constituents under $\mathcal{M}(P_{T_N})$ can be built directly from the respective caplet smile !**

When product modifications to proxy the client requests are not attainable, actual convexity corrections have to be invoked.

For instance, for the valuation of CMS constituents, we may need a quantile map representing the distribution of a swap rate in a T_N forward measure.

Denote $\psi^{\mathcal{M}(A_i)}_{\mathsf{IR},T_i}(\mathsf{s}_i)$ as the risk-neutral density of the swap rate s_i in its natural measure induced by the annuity A_i.

This means that the price of an infinitesimal receiver spread swaption on the forward starting swap rate s_i (i.e. a digital put on the annuity-weighted swap rate) struck at K is given by

$$A_i(0) \cdot \Psi^{\mathcal{M}(A_i)}_{\mathsf{IR},T_i}(K) \quad = \quad A_i(0) \cdot \left[\Phi\left(-d_2\right) + \mathsf{s}_i(0) \cdot \sqrt{T_i} \cdot \varphi\left(d_1\right) \cdot \partial_K \hat{\sigma}_{\mathsf{s}_i}(K)\right] \tag{35}$$

with

$$\Psi^{\mathcal{M}(A_i)\prime}_{\mathsf{IR},T_i}(K) \quad = \quad \psi^{\mathcal{M}(A_i)}_{\mathsf{IR},T_i}(K)\ . \tag{36}$$

For hybrid valuation in the measure $\mathcal{M}(P_{T_N})$, we need to construct a quantile map consistent with the density $\psi_{\mathsf{IR},T_i}^{\mathcal{M}(P_{T_N})}(\mathsf{s}_i)$ which is given by

$$\begin{aligned}\psi_{\mathsf{IR},T_i}^{\mathcal{M}(P_{T_N})}(\mathsf{s}_i) &= \frac{\mathrm{d}\mathcal{M}(P_{T_N})}{\mathrm{d}\mathcal{M}(A_i)} \cdot \psi_{\mathsf{IR},T_i}^{\mathcal{M}(A_i)}(\mathsf{s}_i) \\ &= \frac{A_i(0)}{P_{T_N}(0)} \cdot \frac{P_{T_N}}{A_i} \cdot \psi_{\mathsf{IR},T_i}^{\mathcal{M}(A_i)}(\mathsf{s}_i)\ . \end{aligned} \tag{37}$$

A comparatively straightforward and commonly used approximation for this purpose [Hag03] is

$$\frac{P_j(T_i)}{P_{T_N}(T_i)} \approx (1+\tau \mathsf{s}_i)^{T_N - T_j} \tag{38}$$

which is to be applied to

$$\frac{P_{T_N}}{A_i} = \frac{1}{\sum_{j=i+1}^{N} \tau_j \frac{P_{T_j}}{P_{T_N}}}\ . \tag{39}$$

In other words:

$$\psi_{\mathsf{IR},T_i}^{\mathcal{M}(P_{T_N})}(\mathsf{s}_i) \approx \text{constant} \cdot \frac{\psi_{\mathsf{IR},T_i}^{\mathcal{M}(A_i)}(\mathsf{s}_i)}{\sum_{j=i+1}^{N} \tau_j \left(1+\tau \mathsf{s}_i\right)^{T_N - T_j}} \tag{40}$$

$$\frac{1}{\text{constant}} := \int \frac{\psi_{\mathsf{IR},T_i}^{\mathcal{M}(A_i)}(\mathsf{s})}{\sum_{j=i+1}^{N} \tau_j \left(1+\tau \mathsf{s}\right)^{T_N - T_j}} \, \mathrm{d}s \tag{41}$$

$$\Psi_{\mathsf{IR},T_i}^{\mathcal{M}(P_{T_N})}(\mathsf{s}_i) \approx \text{constant} \cdot \int_{-\infty}^{\mathsf{s}_i} \frac{\psi_{\mathsf{IR},T_i}^{\mathcal{M}(A_i)}(\mathsf{s})}{\sum_{j=i+1}^{N} \tau_j \left(1+\tau \mathsf{s}\right)^{T_N - T_j}} \, \mathrm{d}s \tag{42}$$

Note that all integrals involving $\psi_{\mathsf{IR},T_i}^{\mathcal{M}(A_i)}(\mathsf{s})$ can be changed into integrals over $\Psi_{\mathsf{IR},T_i}^{\mathcal{M}(A_i)}(\mathsf{s})$ by the aid of partial integration.

➜ The quantile map

$$z_{s_i}^{\mathcal{M}(P_{T_N})} \quad \leftrightarrow \quad s_i \tag{43}$$

is implicitly given by

$$\Phi(z_{s_i}) \quad = \quad \Psi_{\mathrm{IR},T_i}^{\mathcal{M}(P_{T_N})}(s_i) \tag{44}$$

and can be set up with monotonicity preserving interpolation algorithms as in the previous sections.

Each point in the quantile map involves a one-dimensional numerical integration to compute the integral on the right hand side of equation (42).

In practice, however, all of the required integrals can be carried out in a single swoop which makes the construction of the constant-yield-to-maturity based swap rate quantile map particularly efficient.

We now have a one-factor representation of the yield curve given by z_{s_i} for each of the observation times T_i.

In order to price hybrid derivatives, we need to construct a convexity corrected quantile map for other involved assets such as equity, FX, commodities, etc.

With the definition

$$\chi_{P_{T_i}}(z_{s_i}) \quad := \quad \frac{\mathrm{d}\mathcal{M}(P_{T_i})}{\mathrm{d}\mathcal{M}(P_{T_N})} \quad \approx \quad \text{constant}^{2} \cdot \left(1 + \tau \cdot \mathsf{s}_i(z_{s_i})\right)^{T_N - T_i} \tag{45}$$

and assuming that the natural measure for the pricing of options in asset class X expiring at T_i is $\mathcal{M}(P_{T_i})$, we have

$$\begin{aligned} \mathsf{E}_{T_i}^{\mathcal{M}(P_{T_N})}\left[\mathbf{1}_{\{x_{T_i}<K\}} \cdot \chi_{P_{T_i}}\right] &= \mathsf{E}_{T_i}^{\mathcal{M}(P_{T_i})}\left[\mathbf{1}_{\{x_{T_i}<K\}}\right] \\ &= \Psi_{\mathsf{X},T_i}^{\mathcal{M}(P_{T_i})}(K) \ . \end{aligned} \tag{46}$$

[2] The normalisation constant is defined by $\mathsf{E}_{T_i}^{\mathcal{M}(P_{T_N})}\left[\chi_{P_{T_i}}\right] = 1$.

Recall that we ultimately seek a quantile map

$$z_{x_{T_i}}^{\mathcal{M}(P_{T_N})} \quad \leftrightarrow \quad x_{T_i} \quad . \tag{47}$$

Assuming the Gaussian copula correlation structure

$$\langle z_{x_{T_i}}, z_{s_i} \rangle \;=\; \varrho_{z_{x_{T_i}} z_{s_i}} \;=\; 2 \cdot \sin\left(\frac{\pi}{6}\rho_{\Phi(z_{x_{T_i}})\,\Phi(z_{s_i})}\right) \tag{48}$$

we can decompose the standard Gaussian variate z_{s_i} into a part that is correlated with $z_{x_{T_i}}$ and an independent standard normal residual ε:

$$z_{s_i} \;:=\; \varrho \cdot z_{x_{T_i}} + \varrho' \cdot \varepsilon \tag{49}$$

with

$$\varrho' \;:=\; \sqrt{1-\varrho^2} \; . \tag{50}$$

This enables us to associate a Gaussian equivalent variable $z_{x_{T_i}}^{\mathcal{M}(P_{T_N})}(K)$ with a given strike K:

$$\mathsf{E}_{T_i}^{\mathcal{M}(P_{T_N})}\left[\mathbf{1}_{\{x_{T_i}<K\}}\cdot\chi_{P_{T_i}}\right] = \int\limits_{z=-\infty}^{z_{x_{T_i}}^{\mathcal{M}(P_{T_N})}(K)}\int\limits_{\varepsilon=-\infty}^{\infty}\chi_{P_{T_i}}(\varrho\cdot z+\varrho'\varepsilon)\cdot\varphi(z)\cdot\varphi(\varepsilon)\,\mathrm{d}z\mathrm{d}\varepsilon \tag{51}$$

Defining the auxiliary function

$$f(z_K) := \int\limits_{-\infty}^{z_K}\int\chi_{P_{T_i}}(\varrho\cdot z+\varrho'\varepsilon)\cdot\varphi(\varepsilon)\,\mathrm{d}\varepsilon\cdot\varphi(z)\,\mathrm{d}z \tag{52}$$

we now invoke the Harrison-Pliska theorem that was used in equation (46)

$$f(z_K) = \mathsf{E}_{T_i}^{\mathcal{M}(P_{T_N})}\left[\mathbf{1}_{\{x_{T_i}<K\}}\cdot\chi_{P_{T_i}}\right] = \mathsf{E}_{T_i}^{\mathcal{M}(P_{T_i})}\left[\mathbf{1}_{\{x_{T_i}<K\}}\right] = \Psi_{\mathsf{X},T_i}^{\mathcal{M}(P_{T_i})}(K) \tag{53}$$

The quantile map

$$z_{x_{T_i}}^{\mathcal{M}(P_{T_N})} \quad \leftrightarrow \quad x_{T_i} \quad .$$

is thus to be constructed as follows:

For any Gaussian equivalent $z_{x_{T_i}}$, compute

$$z_{x_{T_i}} \quad \longrightarrow \quad f(z_{x_{T_i}}) \tag{54}$$

From here, calculate the associated asset price level $x_{T_i} = x_{T_i}(z_{x_{T_i}})$ by numerical inversion[3] of $\Psi_{\mathsf{X},T_i}^{\mathcal{M}(P_{T_i})}$:

$$x_{T_i} \quad = \quad \Psi_{\mathsf{X},T_i}^{\mathcal{M}(P_{T_i})^{-1}}(f(z_{x_{T_i}})) \tag{55}$$

➜ The construction of convexity corrected quantile maps for the individual assets can be done with no more complexity than the construction of the yield curve dynamics!

[3] This is again most efficiently done by tabulating the intermediate values and using a monotonicity preserving inverse interpolation.

It is of course also possible to set up the yield curve dynamics without the constant yield approximation and to use instead fully fledged Markov-functional dynamics.

In that case, the yield curve quantile maps (one for each observation time) ought to be constructed in reverse chronological order.

When calibrating the model to coterminal swaptions, it is efficient to keep the value of the Radon-Nikodym derivative that was used to transform from the swap rate's natural measure to the T_N forward measure and the inverse of the relative numéraire (the T_N maturing zero coupon bond) stored together with the quantile-level associated value of the swap rate itself:

$$\mathsf{s}_i \quad \leftrightarrow \quad z_{\mathsf{s}_i} \quad \leftrightarrow \quad \left(\widetilde{\chi}_{A_i}(z_{\mathsf{s}_i}), \frac{1}{P_{T_N}(T_i|z_{\mathsf{s}_i})} \right) \tag{56}$$

with

$$\widetilde{\chi}_{A_i}(z_{\mathsf{s}_i}) \quad := \quad \frac{A_i(T_i|z_{\mathsf{s}_i})}{P_{T_N}(T_i|z_{\mathsf{s}_i})} \tag{57}$$

The quantile map is again constructed by the aid of the market given risk-neutral probabilities and the fundamental asset pricing theorem

$$\mathsf{E}_{T_i}^{\mathcal{M}(A_i)}\left[\mathbf{1}_{\{s_i<K\}}\right] = \mathsf{E}_{T_i}^{\mathcal{M}(P_{T_N})}\left[\mathbf{1}_{\{s_i<K\}} \cdot \chi_{A_i}\right] \tag{58}$$

$$\Psi_{\mathsf{IR},T_i}^{\mathcal{M}(A_i)}(s_i) = \int_{-\infty}^{z_{s_i}} \chi_{A_i}(z)\varphi(z)\,\mathrm{d}z \tag{59}$$

with

$$\chi_{A_i}(z_{s_i}) := \frac{\widetilde{\chi}_{A_i}(z_{s_i})}{\int \widetilde{\chi}_{A_i}(z)\varphi(z)\,\mathrm{d}z} \tag{60}$$

The construction of the quantile map $s_i \leftrightarrow z_{s_i}$ is again done by computing $\int_{-\infty}^{z_{s_i}} \chi_{A_i}(z)\varphi(z)\,\mathrm{d}z$ for many values of z_{s_i}, and by equating it to the (numerically) invertible function $\Psi_{\mathsf{IR},T_i}^{\mathcal{M}(A_i)}(s_i)$, which is to be solved for the associated value $s_i(z_{s_i})$.

Given the quantile map at a later time,

$$s_{i+1} \quad \leftrightarrow \quad z_{s_{i+1}} \quad \leftrightarrow \quad \left(\widetilde{\chi}_{A_{i+1}}(z_{s_{i+1}}), \frac{1}{P_{T_N}(T_{i+1}|z_{s_{i+1}})}\right) \ , \tag{61}$$

and the temporal copula

$$\left(z_{s_i}, z_{s_{i+1}}\right) \ \sim \ \mathcal{N}\left(0, 0, \left(\begin{smallmatrix} 1 & \varrho_{z_{s_i} z_{s_{i+1}}} \\ \varrho_{z_{s_i} z_{s_{i+1}}} & 1 \end{smallmatrix}\right)\right) \tag{62}$$

the function $\widetilde{\chi}_{A_i}(z_{s_i})$ can be evaluated from

$$\begin{aligned} \widetilde{\chi}_{A_i}(z_{s_i}) &= \frac{A_i(T_i|z_{s_i})}{P_{T_N}(T_i|z_{s_i})} = \frac{A_{i+1}(T_i|z_{s_i})+\tau_i P_{i+1}(T_i|z_{s_i})}{P_{T_N}(T_i|z_{s_i})} \\ &= \mathsf{E}_{T_{i+1}}^{\mathcal{M}(P_{T_N})}\left[\left.\frac{A_{i+1}}{P_{T_N}}\right| z_{s_i}\right] + \tau_i \cdot \mathsf{E}_{T_{i+1}}^{\mathcal{M}(P_{T_N})}\left[\left.\frac{P_{T_{i+1}}}{P_{T_N}}\right| z_{s_i}\right] \\ &= \int \widetilde{\chi}_{A_{i+1}}(z_{s_{i+1}}|z_{s_i})\varphi(\varepsilon)\,\mathrm{d}\varepsilon + \tau_i \cdot \int \frac{\varphi(\varepsilon)}{P_{T_N}(T_{i+1}|(z_{s_{i+1}}|z_{s_i}))}\,\mathrm{d}\varepsilon \end{aligned} \tag{63}$$

which can be done very efficiently by Gauss-Hermite quadrature:

$$\widetilde{\chi}_{A_i}(z_{s_i}) \;=\; \int \widetilde{\chi}_{A_{i+1}}(z_{s_{i+1}}(z_{s_i},\varepsilon))\varphi(\varepsilon)\,\mathrm{d}\varepsilon + \tau_i \int \frac{\varphi(\varepsilon)}{P_{T_N}(T_{i+1}|z_{s_{i+1}}(z_{s_i},\varepsilon))}\,\mathrm{d}\varepsilon \tag{64}$$

with

$$z_{s_{i+1}}(z_{s_i},\varepsilon) \;:=\; z_{s_i}\cdot\varrho_{z_{s_i}z_{s_{i+1}}} + \varepsilon\cdot\sqrt{1-\varrho^2_{z_{s_i}z_{s_{i+1}}}} \tag{65}$$

The inverse numéraire value at T_i conditional on z_{s_i} can be computed from a martingale condition on the numéraire-relative value of the floating leg:

$$\begin{aligned}
s_i A_i &= P_{T_i} - P_{T_{i+1}} + P_{T_{i+1}} - P_{T_N} \\
(s_i A_i)|\,\mathcal{F}_{T_i} &= \big(1 - P_{T_{i+1}} + s_{i+1}(T_i)A_{i+1}\big)\big|\,\mathcal{F}_{T_i} \\
s_i(z_{s_i})\frac{A_i}{P_{T_N}}\bigg|\,z_{s_i} &= \frac{1}{P_{T_N}(T_i|z_{s_i})} - \frac{P_{T_{i+1}}(T_i|z_{s_i})}{P_{T_N}(T_i|z_{s_i})} + \frac{s_{i+1}(T_i|z_{s_i})A_{i+1}(T_i|z_{s_i})}{P_{T_N}(T_i|z_{s_i})}
\end{aligned} \tag{66}$$

$$s_i(z_{s_i})\widetilde{\chi}_{A_i}(z_{s_i}) \;=\; \frac{1}{P_{T_N}(T_i|z_{s_i})} - \mathsf{E}^{\mathcal{M}(P_{T_N})}_{T_{i+1}}\left[\frac{P_{T_{i+1}}}{P_{T_N}}\bigg|\,z_{s_i}\right] + \mathsf{E}^{\mathcal{M}(P_{T_N})}_{T_{i+1}}\left[\frac{s_{i+1}A_{i+1}}{P_{T_N}}\bigg|\,z_{s_i}\right] \tag{67}$$

➜

$$\frac{1}{P_{T_N}(T_i|z_{s_i})} = s_i(z_{s_i})\widetilde{\chi}_{A_i}(z_{s_i}) + \int \frac{\varphi(\varepsilon)}{P_{T_N}(T_{i+1}|z_{s_{i+1}}(z_{s_i},\varepsilon))}\,\mathrm{d}\varepsilon \tag{68}$$

$$- \int s_{i+1}(z_{s_{i+1}}(z_{s_i},\varepsilon)) \cdot \widetilde{\chi}_{A_{i+1}}(z_{s_{i+1}}(z_{s_i},\varepsilon)) \cdot \varphi(\varepsilon)\,\mathrm{d}\varepsilon$$

The construction of convexity corrected quantile maps for the remaining asset classes is not affected by the method chosen for the construction of the yield curve dynamics:

$$\Psi_{\mathsf{X},T_i}^{\mathcal{M}(P_{T_i})}(x_{T_i}) = \int_{-\infty}^{z_{x_{T_i}}} \int \chi_{P_{T_i}}(z_{s_i}(z_x,\varepsilon)) \cdot \varphi(\varepsilon)\,\mathrm{d}\varepsilon \cdot \varphi(z_x)\,\mathrm{d}z_x \tag{69}$$

$$z_{s_i}(z_x,\varepsilon) := z_x \cdot \varrho_{z_{x_{T_i}} z_{s_i}} + \varepsilon \cdot \sqrt{1-\varrho^2_{z_{x_{T_i}} z_{s_i}}} \tag{70}$$

$$x_{T_i} \leftrightarrow z_{x_{T_i}}^{\mathcal{M}(P_{T_N})} \tag{71}$$

VI. Sample structures

- **Best of Asianed equity return, CPI(terminal)/CPI(initial), and fixed return.**

  This constitutes a capital guaranteed note with inflation protection that benefits from any sustained economic upturn.

- **Option on weighted average of Asianed equity, Asianed FX return, Asianed commodity return, and yield curve lift.**

  This kind of deal is also typically wrapped up in a capital guaranteed note for investors. The yield curve lift is here defined as the difference between a CMS rate at expiry minus the same CMS rate at inception.

VII. Mean reversion

Certain underlying investment classes are not (always) well modelled by an auto-correlation structure that is close to that of geometric Brownian motion.

- **Interest rates**
- **Commodities**
- **Inflation**

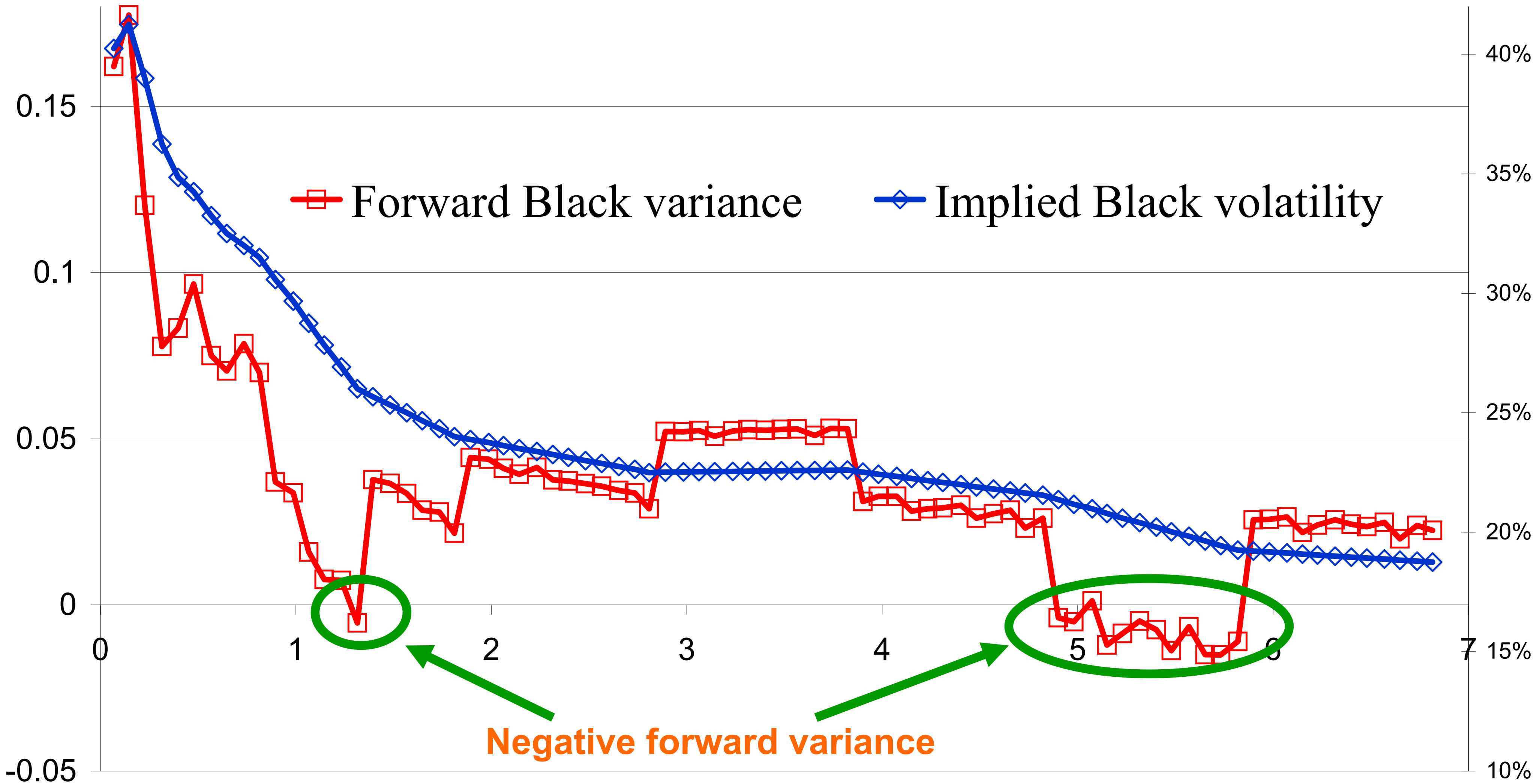


Figure VII.1. Term structure of crude oil implied volatility and associated forward Black variance.

- **When *storing* the investment asset is physically difficult or strictly impossible for fundamental reasons, deformations of the forward curve are possible.**

- **In this case, enforcing (14) is not desirable.**

- **Instead, decorrelation of the curve is desirable for realistic modelling or better repricing of related derivatives.**

The driving process (16) can be replaced by a mean-reverting Gaussian process:

$$\mathrm{d}y(t) = -\varkappa \cdot y(t)\,\mathrm{d}t + \alpha(t)\,\mathrm{d}W$$

The covariances of the driving processes $\#k$ and $\#l$ are given by:

$$\mathrm{Cov}[y_k(t_i), y_l(t_j)] = \mathrm{e}^{-\varkappa_k t_i - \varkappa_l t_j} \int\limits_0^{\min(t_i,t_j)} \mathrm{e}^{(\varkappa_k+\varkappa_l)s} \alpha_k(s)\alpha_l(s)\rho_{kl}(s)\,\mathrm{d}s \tag{72}$$

Note: since the model still always reprices all plain vanilla options correctly by construction, the only effect of the introduction of mean-reversion is a reduction in auto-correlation of the driving process.

➜ The model is de facto identical with a Monte-Carlo version of a multivariate hybrid Markov-functional model.

VIII. Summary

Main points of the ***hybrid Markov functional model***:-

- All plain vanilla options are repriced exactly by construction.
- It is comparatively fast.
- It handles multiple underlyings very naturally.
- For storable investment assets, it should only really be used for weakly path dependent derivatives because it doesn't preserve the risk-neutral drift condition, and then in conjunction with a selected expectation correction method.

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