An asymptotic FX option formula in the cross currency Libor market model

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Abstract

In this article, we introduce analytic approximation formulae for FX options in the Libor market model (LMM). The method to derive the formulae is an asymptotic expansion technique introduced in Kawai [Kaw03]. We first apply the method to the lognormal LMM and lognormal FX model. Then, the method is applied to the displaced diffusion LMM and the displaced diffusion FX model. Some numerical examples show that the derived formulae are sufficiently accurate for practical applications.

1 Introduction

The Libor market model developed by Brace, Gatarek, and Musiela [BGM97], Jamshidian [Jam97], Miltersen, Sandmann, and Sondermann [MSS97] is one of the most popular interest rate models among both academics and practitioners. It is interesting to use the model not only for pure interest rate products but also for long-dated equity/FX products or hybrid products. In this article, we focus on modeling cross currency FX markets using the Libor market model, that is, the cross currency hybrid LMM. The dynamics of the model are a straightforward extension of the standard LMM formulation, and considerations regarding the specific choice of FX state variable (spot, forward, rolling spot) are discussed in the literature [Sch02]. Whilst a variety of accurate and efficient approximations for vanilla swaption pricing, and thus model calibration are available [JR00, HW00, Kaw02, Kaw03], very little has been published with respect to vanilla FX option approximations in cross-currency FX/interest rate models. The most notable exceptions are the very recent works by Osajima [Osa06] for a Gaussian forward rate stochastic volatility setup, and [AM06a, AM06b] for a cross-currency Libor market model without explicit skew on FX and interest rates. Since European FX options are the most important hedge instruments for the cross-currency exposure of FX/interest rate contracts, and since the pricing of vanilla options using Monte Carlo simulations for calibration purposes can be rather cumbersome, analytical approximations for FX plain vanilla option prices are highly desirable. Using an asymptotic expansion method introduced in Kawai [Kaw03], we derive an analytic approximation formula for European FX options in the lognormal[1] cross currency hybrid LMM. Then, we extend the method to the displaced diffusion cross currency hybrid LMM.

In the next section, we briefly review our model setup. In section 3, we present the FX option formula directly in terms of a strike-dependent equivalent Black implied volatility for the lognormal LMM. Then, in section 4, we extend our results to the extended cross-currency LMM in which the domestic rates, the foreign rates, and the FX spot process all are governed by a skew-generating local volatility process known as displaced diffusion [Rub83]. Following that, we show some numerical results for comparison of analytics and Monte Carlo simulations. Finally, we conclude.

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[1] Note that we use the term *lognormal* often in an approximate sense. Strictly speaking, due to stochastic drift terms, marginal distributions of forward interest and FX rates are not exactly lognormal.

2 The cross-currency hybrid LMM

In the lognormal multi-currency LMM, the spot FX rate Q, the domestic forward Libor rates f_i^{D} and the foreign forward Libor rates f_i^{F} evolve lognormally according to the following stochastic differential equations in the domestic T_N forward measure.

- The spot FX rate

$$\frac{\mathrm{d}Q}{Q} = \mu_Q \,\mathrm{d}t + \sigma_Q \,\mathrm{d}\widetilde{W}_Q \;, \qquad \text{with} \qquad \mu_Q = r^{\mathsf{D}} - r^{\mathsf{F}} - \sigma_Q \sum_{k=0}^{N-1} \frac{f_k^{\mathsf{D}} \tau_k^{\mathsf{D}}}{1 + f_k^{\mathsf{D}} \tau_k^{\mathsf{D}}} \sigma_k^{\mathsf{D}} \varrho_{Q,f_k^{\mathsf{D}}} \;. \tag{2.1}$$

  Note that the domestic and foreign short rates r^{D} and r^{F} are herein only auxiliary concepts which we can convert into discrete forward rates once we integrate over time. Alternatively, we can consider forward FX rates in their own natural measure which removes any drift terms altogether, and this is indeed the approach we take in the derivation of our formulae in the appendix.

- The domestic forward Libor rates

$$\frac{\mathrm{d}f_i^{\mathsf{D}}}{f_i^{\mathsf{D}}} = \mu_i^{\mathsf{D}} \,\mathrm{d}t + \sigma_i^{\mathsf{D}} \,\mathrm{d}\widetilde{W}_i^{\mathsf{D}} \;, \qquad \text{with} \qquad \mu_i^{\mathsf{D}} = -\sigma_i^{\mathsf{D}} \sum_{k=i+1}^{N-1} \frac{f_k^{\mathsf{D}} \tau_k^{\mathsf{D}}}{1 + f_k^{\mathsf{D}} \tau_k^{\mathsf{D}}} \sigma_k^{\mathsf{D}} \varrho_{f_i^{\mathsf{D}},f_k^{\mathsf{D}}} \;. \tag{2.2}$$

- The foreign forward Libor rates

$$\frac{\mathrm{d}f_i^{\mathsf{F}}}{f_i^{\mathsf{F}}} = \mu_i^{\mathsf{F}} \,\mathrm{d}t + \sigma_i^{\mathsf{F}} \,\mathrm{d}\widetilde{W}_i^{\mathsf{F}} \;, \tag{2.3}$$

  with

$$\mu_i^{\mathsf{F}} = \sigma_i^{\mathsf{F}} \left(-\sigma_Q \varrho_{Q,f_i^{\mathsf{F}}} + \sum_{k=0}^{i} \frac{f_k^{\mathsf{F}} \tau_k^{\mathsf{F}}}{1 + f_k^{\mathsf{F}} \tau_k^{\mathsf{F}}} \sigma_k^{\mathsf{F}} \varrho_{f_i^{\mathsf{F}},f_k^{\mathsf{F}}} - \sum_{k=0}^{N-1} \frac{f_k^{\mathsf{D}} \tau_k^{\mathsf{D}}}{1 + f_k^{\mathsf{D}} \tau_k^{\mathsf{D}}} \sigma_k^{\mathsf{D}} \varrho_{f_i^{\mathsf{D}},f_k^{\mathsf{F}}} \right) \;. \tag{2.4}$$

Correlations are incorporated by the fact that the individual standard Wiener processes in equation (2.1), (2.2) and (2.3) satisfy

$$\begin{gathered} \mathsf{E}\left[\mathrm{d}\widetilde{W}_Q \,\mathrm{d}\widetilde{W}_i^{\mathsf{D}}\right] = \varrho_{Q,f_i^{\mathsf{D}}} \mathrm{d}t \;, \quad \mathsf{E}\left[\mathrm{d}\widetilde{W}_Q \,\mathrm{d}\widetilde{W}_i^{\mathsf{F}}\right] = \varrho_{Q,f_i^{\mathsf{F}}} \mathrm{d}t \;, \quad \mathsf{E}\left[\mathrm{d}\widetilde{W}_i^{\mathsf{D}} \,\mathrm{d}\widetilde{W}_j^{\mathsf{D}}\right] = \varrho_{f_i^{\mathsf{D}},f_j^{\mathsf{D}}} \mathrm{d}t \;, \\ \mathsf{E}\left[\mathrm{d}\widetilde{W}_i^{\mathsf{F}} \,\mathrm{d}\widetilde{W}_j^{\mathsf{F}}\right] = \varrho_{f_i^{\mathsf{F}},f_j^{\mathsf{F}}} \mathrm{d}t \;, \quad \mathsf{E}\left[\mathrm{d}\widetilde{W}_i^{\mathsf{D}} \,\mathrm{d}\widetilde{W}_j^{\mathsf{F}}\right] = \varrho_{f_i^{\mathsf{D}},f_j^{\mathsf{F}}} \mathrm{d}t \;. \end{gathered} \tag{2.5}$$

3 The asymptotic FX option formula

Based on the dynamics of the FX rate and interest rates described above, we can obtain an analytic approximate European FX option formula using the asymptotic expansion method. The detailed derivation of the formula can be found in appendix A. Indeed, the formula is in the form of the European Black option formula and the resulting Black volatility contains all the information of stochasticity of interest rates.

Let $Q_{T_N}(0)$ be the forward FX rate for maturity T_N, that is,

$$Q_{T_N}(0) \;=\; Q(0) \cdot \frac{P_{T_N}^{\mathsf{F}}(0)}{P_{T_N}^{\mathsf{D}}(0)} \;, \tag{3.1}$$

where $P_{T_N}^{\mathsf{D}}(0)$ and $P_{T_N}^{\mathsf{F}}(0)$ are domestic and foreign T_N discount factors respectively as seen at time 0, i.e. today, though, we shall in the following often omit the explicit mentioning "(0)" for the sake of brevity. Then, approximate European FX option prices with strike K and maturity T_N are given by the following formulae.

- Call option

$$P_{T_N}^{\mathsf{D}} \cdot \left[Q_{T_N} \cdot \Phi(h) - K \cdot \Phi(h - \sigma_{\text{Black}} \sqrt{T_N}) \right] \;, \tag{3.2}$$

- Put option

$$P^{\mathsf{D}}_{T_N} \cdot \left[K \cdot \Phi(-h + \sigma_{\text{Black}}\sqrt{T_N}) - Q_{T_N} \cdot \Phi(-h)\right] \ , \tag{3.3}$$

where

$$h = \frac{\ln\left(Q_{T_N}/K\right) + \frac{1}{2}\sigma^2_{\text{Black}} T_N}{\sigma_{\text{Black}}\sqrt{T_N}}, \tag{3.4}$$

$\Phi(\cdot)$ is the cumulative standard normal distribution function, and

$$\sigma^2_{\text{Black}} \ = \ \frac{v_1}{T_N}\left[1 + \left(\frac{1}{Q_{T_N}} - 2c_1\right) g_0 + \left(c_2 + \frac{11}{12Q^2_{T_N}} - \frac{2c_1}{Q_{T_N}}\right) g_0^2 + \left(c_3 + \frac{1}{12Q^2_{T_N}} + 2c_4\right) Q^2_{T_N} v_1\right] \ . \tag{3.5}$$

Also,

$$g_0 = Q_{T_N} - K \ . \tag{3.6}$$

Before we provide the details of the constants v_1, c_1, c_2, c_3 and c_4, we need to make some auxiliary definitions. First, consider the integrated covariances between the processes:

$$\begin{aligned}
c_{Q,Q} &= \int_0^{T_N} \sigma_Q(u)\sigma_Q(u)\,\mathrm{d}u & c_{Q,f^{\mathsf{D}}_k} &= \int_0^{T_N} \sigma_Q(u)\sigma^{\mathsf{D}}_k(u)\varrho_{Q,f^{\mathsf{D}}_k}(u)\,\mathrm{d}u \\
c_{Q,f^{\mathsf{F}}_k} &= \int_0^{T_N} \sigma_Q(u)\sigma^{\mathsf{F}}_k(u)\varrho_{Q,f^{\mathsf{F}}_k}(u)\,\mathrm{d}u & c_{f^{\mathsf{D}}_k,f^{\mathsf{D}}_j} &= \int_0^{T_N} \sigma^{\mathsf{D}}_k(u)\sigma^{\mathsf{D}}_j(u)\varrho_{f^{\mathsf{D}}_k,f^{\mathsf{D}}_j}(u)\,\mathrm{d}u \\
c_{f^{\mathsf{F}}_k,f^{\mathsf{F}}_j} &= \int_0^{T_N} \sigma^{\mathsf{F}}_k(u)\sigma^{\mathsf{F}}_j(u)\varrho_{f^{\mathsf{F}}_k,f^{\mathsf{F}}_j}(u)\,\mathrm{d}u & c_{f^{\mathsf{D}}_k,f^{\mathsf{F}}_j} &= \int_0^{T_N} \sigma^{\mathsf{D}}_k(u)\sigma^{\mathsf{F}}_j(u)\varrho_{f^{\mathsf{D}}_k,f^{\mathsf{F}}_j}(u)\,\mathrm{d}u
\end{aligned} \tag{3.7}$$

Second, define the weights of initial forward Libor rates as follows.

$$w^{\mathsf{D}}_k = \frac{f^{\mathsf{D}}_k(0)\tau^{\mathsf{D}}_k}{1 + f^{\mathsf{D}}_k(0)\tau^{\mathsf{D}}_k} \ , \quad w^{\mathsf{F}}_k = \frac{f^{\mathsf{F}}_k(0)\tau^{\mathsf{F}}_k}{1 + f^{\mathsf{F}}_k(0)\tau^{\mathsf{F}}_k} \ . \tag{3.8}$$

$$y^{\mathsf{D}}_k = \frac{f^{\mathsf{D}}_k(0)\tau^{\mathsf{D}}_k}{\left(1 + f^{\mathsf{D}}_k(0)\tau^{\mathsf{D}}_k\right)^2} \ , \quad y^{\mathsf{F}}_k = \frac{f^{\mathsf{F}}_k(0)\tau^{\mathsf{F}}_k}{\left(1 + f^{\mathsf{F}}_k(0)\tau^{\mathsf{F}}_k\right)^2} \ . \tag{3.9}$$

Then define constant v_1, $v^{\mathsf{D}}_{2,k}$, $v^{\mathsf{F}}_{2,k}$, $v^{\mathsf{D}}_{3,k}$ and $v^{\mathsf{F}}_{3,k}$ as follows.

$$v_1 \ = c_{Q,Q} + \sum_{j,k=0}^{N-1}\left(w^{\mathsf{D}}_k w^{\mathsf{D}}_j c_{f^{\mathsf{D}}_k,f^{\mathsf{D}}_j} + w^{\mathsf{F}}_k w^{\mathsf{F}}_j c_{f^{\mathsf{F}}_k,f^{\mathsf{F}}_j} - 2w^{\mathsf{D}}_k w^{\mathsf{F}}_j c_{f^{\mathsf{D}}_k,f^{\mathsf{F}}_j}\right) + 2\sum_{k=0}^{N-1}\left(w^{\mathsf{D}}_k c_{Q,f^{\mathsf{D}}_k} - w^{\mathsf{F}}_k c_{Q,f^{\mathsf{F}}_k}\right) \tag{3.10}$$

$$v^{\mathsf{D}}_{2,k} \ = c_{Q,f^{\mathsf{D}}_k} + \sum_{j=0}^{N-1}\left(w^{\mathsf{D}}_j c_{f^{\mathsf{D}}_k,f^{\mathsf{D}}_j} - w^{\mathsf{F}}_j c_{f^{\mathsf{D}}_k,f^{\mathsf{F}}_j}\right) \qquad v^{\mathsf{F}}_{2,k} \ = \ c_{Q,f^{\mathsf{F}}_k} + \sum_{j=0}^{N-1}\left(w^{\mathsf{D}}_j c_{f^{\mathsf{F}}_k,f^{\mathsf{D}}_j} - w^{\mathsf{F}}_j c_{f^{\mathsf{F}}_k,f^{\mathsf{F}}_j}\right) \tag{3.11}$$

$$v^{\mathsf{D}}_{3,k} \ = \ -\sum_{j=k+1}^{N-1} w^{\mathsf{D}}_j c_{f^{\mathsf{D}}_k,f^{\mathsf{D}}_j} \qquad v^{\mathsf{F}}_{3,k} \ = \ -c_{Q,f^{\mathsf{F}}_k} + \sum_{j=0}^{k} w^{\mathsf{F}}_j c_{f^{\mathsf{F}}_k,f^{\mathsf{F}}_j} - \sum_{j=0}^{N-1} w^{\mathsf{D}}_j c_{f^{\mathsf{F}}_k,f^{\mathsf{D}}_j} \tag{3.12}$$

Now, the constants c_1, c_2, c_3, and c_4 are given as follows.

$$c_1 \ = \ \frac{1}{2Q_{T_N}v_1^2}\left[v_1^2 + \sum_{k=0}^{N-1}\left(y^{\mathsf{D}}_k \ {v^{\mathsf{D}}_{2,k}}^2 - y^{\mathsf{F}}_k \ {v^{\mathsf{F}}_{2,k}}^2\right)\right] \ , \tag{3.13}$$

$$c_2 \ = \ -5c_1^2 + 2g_1 + 2g_3 + d_1 \ , \tag{3.14}$$

$$c_3 \;=\; 3c_1^2 - 2g_1 + 2g_2 - 2g_3 - d_1 + d_2 \;, \tag{3.15}$$

$$c_4 \;=\; \frac{1}{2Q_{T_N}^2 v_1^2} \sum_{k=0}^{N-1} \left(y_k^{\mathsf{D}} v_{2,k}^{\mathsf{D}} v_{3,k}^{\mathsf{D}} - y_k^{\mathsf{F}} v_{2,k}^{\mathsf{F}} v_{3,k}^{\mathsf{F}} \right) \;, \tag{3.16}$$

with

$$g_1 = \frac{1}{6Q_{T_N}^2} \;, \tag{3.17}$$

$$g_2 = \frac{1}{2Q_{T_N}^2 v_1^2} \sum_{k=0}^{N-1} \left(y_k^{\mathsf{D}} \; v_{2,k}^{\mathsf{D}\,2} - y_k^{\mathsf{F}} \; v_{2,k}^{\mathsf{F}\,2} \right) \;, \tag{3.18}$$

$$g_3 = \frac{1}{6Q_{T_N}^2 v_1^3} \sum_{k=0}^{N-1} \left(y_k^{\mathsf{D}} \; v_{2,k}^{\mathsf{D}\,2} \left(3v_1 + v_{2,k}^{\mathsf{D}}\right) - y_k^{\mathsf{F}} \; v_{2,k}^{\mathsf{F}\,2} \left(3v_1 + v_{2,k}^{\mathsf{F}}\right) \right) \;, \tag{3.19}$$

$$\begin{aligned} d_1 \;=\; \frac{1}{Q_{T_N}^2 v_1^3} \Bigg[& v_1^3 + 2v_1 \sum_{k=0}^{N-1} \left(y_k^{\mathsf{D}} \; v_{2,k}^{\mathsf{D}\,2} - y_k^{\mathsf{F}} \; v_{2,k}^{\mathsf{F}\,2} \right) \\ & + \sum_{j,k=0}^{N-1} \left(y_k^{\mathsf{D}} y_j^{\mathsf{D}} v_{2,k}^{\mathsf{D}} v_{2,j}^{\mathsf{D}} c_{f_k^{\mathsf{D}}, f_j^{\mathsf{D}}} + y_k^{\mathsf{F}} y_j^{\mathsf{F}} v_{2,k}^{\mathsf{F}} v_{2,j}^{\mathsf{F}} c_{f_k^{\mathsf{F}}, f_j^{\mathsf{F}}} - 2 y_k^{\mathsf{D}} y_j^{\mathsf{F}} v_{2,k}^{\mathsf{D}} v_{2,j}^{\mathsf{F}} c_{f_k^{\mathsf{D}}, f_j^{\mathsf{F}}} \right) \Bigg] \;, \end{aligned} \tag{3.20}$$

and

$$d_2 \;=\; \frac{1}{2Q_{T_N}^2 v_1^2} \left[v_1^2 + \sum_{k=0}^{N-1} \left(y_k^{\mathsf{D}} \; v_{2,k}^{\mathsf{D}\,2} + y_k^{\mathsf{F}} \; v_{2,k}^{\mathsf{F}\,2} \right) + \sum_{j,k=0}^{N-1} \left(y_k^{\mathsf{D}} y_j^{\mathsf{D}} c_{f_k^{\mathsf{D}}, f_j^{\mathsf{D}}}^2 + y_k^{\mathsf{F}} y_j^{\mathsf{F}} c_{f_k^{\mathsf{F}}, f_j^{\mathsf{F}}}^2 - 2 y_k^{\mathsf{D}} y_j^{\mathsf{F}} c_{f_k^{\mathsf{D}}, f_j^{\mathsf{F}}}^2 \right) \right] \;. \tag{3.21}$$

4 The FX option formula with skew

This section extends the FX option formula when the spot FX and interest rates follow displaced diffusion processes. We assume the following dynamics for the spot FX rate, the domestic forward Libor rates and the foreign forward Libor rates.

- The spot FX rate

$$\frac{\mathrm{d}\left(Q + s_Q\right)}{Q + s_Q} = \mu_Q \,\mathrm{d}t + \sigma_Q \,\mathrm{d}\widetilde{W}_Q \;, \text{ with } \quad \mu_Q = \frac{Q}{Q + s_Q} \left(r^{\mathsf{D}} - r^{\mathsf{F}} \right) - \sigma_Q \sum_{k=0}^{N-1} \frac{\left(f_k^{\mathsf{D}} + s_k^{\mathsf{D}}\right)\tau_k^{\mathsf{D}}}{1 + f_k^{\mathsf{D}}\tau_k^{\mathsf{D}}} \sigma_k^{\mathsf{D}} \varrho_{Q, f_k^{\mathsf{D}}} \;. \tag{4.1}$$

- The domestic forward Libor rates

$$\frac{\mathrm{d}\left(f_i^{\mathsf{D}} + s_i^{\mathsf{D}}\right)}{f_i^{\mathsf{D}} + s_i^{\mathsf{D}}} = \mu_i^{\mathsf{D}} \,\mathrm{d}t + \sigma_i^{\mathsf{D}} \,\mathrm{d}\widetilde{W}_i \;, \qquad \text{with} \qquad \mu_i^{\mathsf{D}} = -\sigma_i^{\mathsf{D}} \sum_{k=i+1}^{N-1} \frac{\left(f_k^{\mathsf{D}} + s_k^{\mathsf{D}}\right)\tau_k^{\mathsf{D}}}{1 + f_k^{\mathsf{D}}\tau_k^{\mathsf{D}}} \sigma_k^{\mathsf{D}} \varrho_{f_i^{\mathsf{D}}, f_k^{\mathsf{D}}} \;. \tag{4.2}$$

- The foreign forward Libor rates

$$\frac{\mathrm{d}\left(f_i^{\mathsf{F}} + s_i^{\mathsf{F}}\right)}{f_i^{\mathsf{F}} + s_i^{\mathsf{F}}} = \mu_i^{\mathsf{F}} \,\mathrm{d}t + \sigma_i^{\mathsf{F}} \,\mathrm{d}\widetilde{W}_i^{\mathsf{F}} \;, \tag{4.3}$$

with

$$\mu_i^{\mathsf{F}} = \sigma_i^{\mathsf{F}} \left(-\frac{Q + s_Q}{Q} \sigma_Q \varrho_{Q, f_i^{\mathsf{F}}} + \sum_{k=0}^{i} \frac{\left(f_k^{\mathsf{F}} + s_k^{\mathsf{F}}\right)\tau_k^{\mathsf{F}}}{1 + f_k^{\mathsf{F}}\tau_k^{\mathsf{F}}} \sigma_k^{\mathsf{F}} \varrho_{f_i^{\mathsf{F}}, f_k^{\mathsf{F}}} - \sum_{k=0}^{N-1} \frac{\left(f_k^{\mathsf{D}} + s_k^{\mathsf{D}}\right)\tau_k^{\mathsf{D}}}{1 + f_k^{\mathsf{D}}\tau_k^{\mathsf{D}}} \sigma_k^{\mathsf{D}} \varrho_{f_i^{\mathsf{D}}, f_k^{\mathsf{F}}} \right) \;. \tag{4.4}$$

Then, in this extended framework the approximate European FX option prices with strike K and maturity T_N are given by the following formulae.

- Call option

$$P^{\mathsf{D}}_{T_N} \cdot \left(Q^{\text{Disp}}_{T_N} \cdot \Phi(h) - K^{\text{Disp}} \cdot \Phi(h - \sigma_{\text{Disp}} \sqrt{T_N}) \right) \ , \tag{4.5}$$

- Put option

$$P^{\mathsf{D}}_{T_N} \cdot \left(K^{\text{Disp}} \cdot \Phi(-h + \sigma_{\text{Disp}} \sqrt{T_N}) - Q^{\text{Disp}}_{T_N} \cdot \Phi(-h) \right) \ , \tag{4.6}$$

where

$$h = \frac{\ln\left(Q^{\text{Disp}}_{T_N} / K^{\text{Disp}}\right) + \frac{1}{2}\sigma^2_{\text{Disp}} T_N}{\sigma_{\text{Disp}} \sqrt{T_N}}, \tag{4.7}$$

and

$$\sigma^2_{\text{Disp}} \ = \ \beta^2 \frac{v_1}{T_N} \left[1 + \left(\frac{1}{Q^{\text{Disp}}_{T_N}} - 2c_1 \right) g_0 + \left(c_2 + \frac{11}{12\, {Q^{\text{Disp}}_{T_N}}^2} - \frac{2c_1}{Q^{\text{Disp}}_{T_N}} \right) g_0^2 + \left(c_3 + \frac{1}{12\, {Q^{\text{Disp}}_{T_N}}^2} + 2c_4 \right) Q^2_{T_N} v_1 \right] . \tag{4.8}$$

Here

$$g_0 = Q_{T_N} - K \ , \quad Q^{\text{Disp}}_{T_N} = Q_{T_N} + s_Q \ , \quad K^{\text{Disp}} = K + s_Q \ , \quad \text{and} \quad \beta = Q_{T_N} / Q^{\text{Disp}}_{T_N} \ . \tag{4.9}$$

Furthermore, we have new constants v_1, c_1, c_2, c_3 and c_4. Let us first define the *displacement correction factor* as

$$\gamma = 1 + \frac{1}{2} s_Q \left(\frac{1}{Q_{T_N}} + \frac{1}{Q} \right) \ . \tag{4.10}$$

Second, define the weights of initial forward Libor rates as follows.

$$w^{\mathsf{D}}_k \ = \frac{\left(f^{\mathsf{D}}_k(0) + s^{\mathsf{D}}_k\right) \tau^{\mathsf{D}}_k}{1 + f^{\mathsf{D}}_k(0)\tau^{\mathsf{D}}_k} \qquad w^{\mathsf{F}}_k \ = \frac{\left(f^{\mathsf{F}}_k(0) + s^{\mathsf{F}}_k\right) \tau^{\mathsf{F}}_k}{1 + f^{\mathsf{F}}_k(0)\tau^{\mathsf{F}}_k} \tag{4.11}$$

$$y^{\mathsf{D}}_k \ = \frac{\left(f^{\mathsf{D}}_k(0) + s^{\mathsf{D}}_k\right) \left(\tau^{\mathsf{D}}_k - s^{\mathsf{D}}_k\, {\tau^{\mathsf{D}}_k}^2\right)}{\left(1 + f^{\mathsf{D}}_k(0)\tau^{\mathsf{D}}_k\right)^2} \qquad y^{\mathsf{F}}_k \ = \frac{\left(f^{\mathsf{F}}_k(0) + s^{\mathsf{F}}_k\right) \left(\tau^{\mathsf{F}}_k - s^{\mathsf{F}}_k\, {\tau^{\mathsf{F}}_k}^2\right)}{\left(1 + f^{\mathsf{F}}_k(0)\tau^{\mathsf{F}}_k\right)^2} \tag{4.12}$$

Then, define the constants v_1, $v^{\mathsf{D}}_{2,k}$, $v^{\mathsf{F}}_{2,k}$, $v^{\mathsf{D}}_{3,k}$, $v^{\mathsf{F}}_{3,k}$, v_4 and v_5 as follows.

$$v_1 \ = \gamma^2 c_{Q,Q} + \sum_{j,k=0}^{N-1} \left(w^{\mathsf{D}}_k w^{\mathsf{D}}_j c_{f^{\mathsf{D}}_k, f^{\mathsf{D}}_j} + w^{\mathsf{F}}_k w^{\mathsf{F}}_j c_{f^{\mathsf{F}}_k, f^{\mathsf{F}}_j} - 2 w^{\mathsf{D}}_k w^{\mathsf{F}}_j c_{f^{\mathsf{D}}_k, f^{\mathsf{F}}_j} \right) + 2\gamma \sum_{k=0}^{N-1} \left(w^{\mathsf{D}}_k c_{Q, f^{\mathsf{D}}_k} - w^{\mathsf{F}}_k c_{Q, f^{\mathsf{F}}_k} \right) \tag{4.13}$$

$$v^{\mathsf{D}}_{2,k} \ = \gamma c_{Q, f^{\mathsf{D}}_k} + \sum_{j=0}^{N-1} \left(w^{\mathsf{D}}_j c_{f^{\mathsf{D}}_k, f^{\mathsf{D}}_j} - w^{\mathsf{F}}_j c_{f^{\mathsf{D}}_k, f^{\mathsf{F}}_j} \right) \qquad v^{\mathsf{F}}_{2,k} \ = \ \gamma c_{Q, f^{\mathsf{F}}_k} + \sum_{j=0}^{N-1} \left(w^{\mathsf{D}}_j c_{f^{\mathsf{F}}_k, f^{\mathsf{D}}_j} - w^{\mathsf{F}}_j c_{f^{\mathsf{F}}_k, f^{\mathsf{F}}_j} \right) \tag{4.14}$$

$$v^{\mathsf{D}}_{3,k} \ = \ - \sum_{j=k+1}^{N-1} w^{\mathsf{D}}_j c_{f^{\mathsf{D}}_k, f^{\mathsf{D}}_j} \qquad v^{\mathsf{F}}_{3,k} \ = \ -\gamma c_{Q, f^{\mathsf{F}}_k} + \sum_{j=0}^{k} w^{\mathsf{F}}_j c_{f^{\mathsf{F}}_k, f^{\mathsf{F}}_j} - \sum_{j=0}^{N-1} w^{\mathsf{D}}_j c_{f^{\mathsf{F}}_k, f^{\mathsf{D}}_j} \tag{4.15}$$

$$v_4 \ = \gamma c_{Q,Q} + \sum_{j=0}^{N-1} \left(w^{\mathsf{D}}_j c_{Q, f^{\mathsf{D}}_j} - w^{\mathsf{F}}_j c_{Q, f^{\mathsf{F}}_j} \right) \qquad v_5 \ = \ - \sum_{j=0}^{N-1} w^{\mathsf{D}}_j c_{Q, f^{\mathsf{D}}_j} \ . \tag{4.16}$$

Now, the constants c_1, c_2, c_3 and c_4 are given by:

$$c_1 \ = \frac{1}{2 Q_{T_N} v_1^2} \left[v_1^2 - \gamma\, (\gamma - 1)\, v_4^2 + \sum_{k=0}^{N-1} \left(y^{\mathsf{D}}_k \ {v^{\mathsf{D}}_{2,k}}^2 - y^{\mathsf{F}}_k \ {v^{\mathsf{F}}_{2,k}}^2 \right) \right] \ , \tag{4.17}$$

$$c_2 \ = \ - 5 c_1^2 + 2 g_1 + 2 g_3 + d_1 \ , \tag{4.18}$$

$$c_3 = 3c_1^2 - 2g_1 + 2g_2 - 2g_3 - d_1 + d_2 \ , \tag{4.19}$$

$$c_4 = \frac{1}{2Q_{T_N}^2 v_1^2}\left[\gamma\left(\gamma-1\right)v_4 v_5 + \sum_{k=0}^{N-1}\left(y_k^{\mathsf{D}} v_{2,k}^{\mathsf{D}} v_{3,k}^{\mathsf{D}} - y_k^{\mathsf{F}} v_{2,k}^{\mathsf{F}} v_{3,k}^{\mathsf{F}}\right)\right] \ , \tag{4.20}$$

with

$$g_1 = \frac{1}{6Q_{T_N}^2} \ , \tag{4.21}$$

$$g_2 = \frac{1}{2Q_{T_N}^2 v_1^2}\left[-\gamma\left(\gamma-1\right)v_4^2 + \gamma^2\left(\gamma-1\right)v_4 c_{Q,Q} + \sum_{k=0}^{N-1}\left(y_k^{\mathsf{D}}\ {v_{2,k}^{\mathsf{D}}}^2 - y_k^{\mathsf{F}}\ {v_{2,k}^{\mathsf{F}}}^2\right)\right] \ , \tag{4.22}$$

$$\begin{aligned} g_3 = \frac{1}{6Q_{T_N}^2 v_1^3}\Bigg[&-3\gamma\left(\gamma-1\right)v_1 v_4^2 + \gamma\left(2\gamma-1\right)\left(\gamma-1\right)v_4^3 \\ &+ \sum_{k=0}^{N-1}\left(y_k^{\mathsf{D}}\ {v_{2,k}^{\mathsf{D}}}^2\left(3v_1 + v_{2,k}^{\mathsf{D}}\right) - y_k^{\mathsf{F}}\ {v_{2,k}^{\mathsf{F}}}^2\left(3v_1 + v_{2,k}^{\mathsf{F}}\right)\right)\Bigg] \ , \end{aligned} \tag{4.23}$$

$$\begin{aligned} d_1 = \frac{1}{Q_{T_N}^2 v_1^3}\Bigg[&v_1^3 + \gamma^2\left(\gamma-1\right)^2 v_4^2 c_{Q,Q} - 2\gamma\left(\gamma-1\right)v_1 v_4^2 \\ &+ 2\sum_{k=0}^{N-1}\left(y_k^{\mathsf{D}} v_1\ {v_{2,k}^{\mathsf{D}}}^2 - y_k^{\mathsf{F}} v_1\ {v_{2,k}^{\mathsf{F}}}^2\right) \qquad (4.24) \\ &- 2\gamma\left(\gamma-1\right)\sum_{k=0}^{N-1}\left(y_k^{\mathsf{D}} v_{2,k}^{\mathsf{D}} v_4 c_{Q,f_k^{\mathsf{D}}} - y_k^{\mathsf{F}} v_{2,k}^{\mathsf{F}} v_4 c_{Q,f_k^{\mathsf{F}}}\right) \\ &+ \sum_{j,k=0}^{N-1}\left(y_k^{\mathsf{D}} y_j^{\mathsf{D}} v_{2,k}^{\mathsf{D}} v_{2,j}^{\mathsf{D}} c_{f_k^{\mathsf{D}},f_j^{\mathsf{D}}} + y_k^{\mathsf{F}} y_j^{\mathsf{F}} v_{2,k}^{\mathsf{F}} v_{2,j}^{\mathsf{F}} c_{f_k^{\mathsf{F}},f_j^{\mathsf{F}}} - 2y_k^{\mathsf{D}} y_j^{\mathsf{F}} v_{2,k}^{\mathsf{D}} v_{2,j}^{\mathsf{F}} c_{f_k^{\mathsf{D}},f_j^{\mathsf{F}}}\right)\Bigg] \ , \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} d_2 = \frac{1}{2Q_{T_N}^2 v_1^2}\Bigg[&v_1^2 + \gamma^2\left(\gamma-1\right)^2 c_{Q,Q}^2 - 2\gamma\left(\gamma-1\right)v_4^2 \\ &+ \sum_{k=0}^{N-1}\left(y_k^{\mathsf{D}}\ {v_{2,k}^{\mathsf{D}}}^2 - y_k^{\mathsf{F}}\ {v_{2,k}^{\mathsf{F}}}^2\right) - \gamma\left(\gamma-1\right)\sum_{k=0}^{N-1}\left(y_k^{\mathsf{D}} c_{Q,f_k^{\mathsf{D}}}^2 - y_k^{\mathsf{F}} c_{Q,f_k^{\mathsf{F}}}^2\right) \\ &+ \sum_{j,k=0}^{N-1}\left(y_k^{\mathsf{D}} y_j^{\mathsf{D}} c_{f_k^{\mathsf{D}},f_j^{\mathsf{D}}}^2 + y_k^{\mathsf{F}} y_j^{\mathsf{F}} c_{f_k^{\mathsf{F}},f_j^{\mathsf{F}}}^2 - 2y_k^{\mathsf{D}} y_j^{\mathsf{F}} c_{f_k^{\mathsf{D}},f_j^{\mathsf{F}}}^2\right)\Bigg] \ . \end{aligned} \tag{4.26}$$

5 Numerical results

In this section, we present some numerical results showing the accuracy of our analytic approximations by comparing with Monte Carlo valuations.

In figure 1, we show the FX implied volatility profile for different maturities from three months to fifteen years. Both the domestic and the foreign currency's interest rates were set to be flat at 5%, and the yield curves were individually driven by a single factor, with equal levels of volatility. All displacement coefficients $s_{(\cdot)}$ were set to zero. As we can see, this kind of symmetric setup gives rise to a moderate symmetric smile that is generated solely by the fact that the FX rate's instantaneous dynamics have a drift component that is stochastic in its own right in a non-Gaussian fashion. In figure 2, we repeated the same experiment with

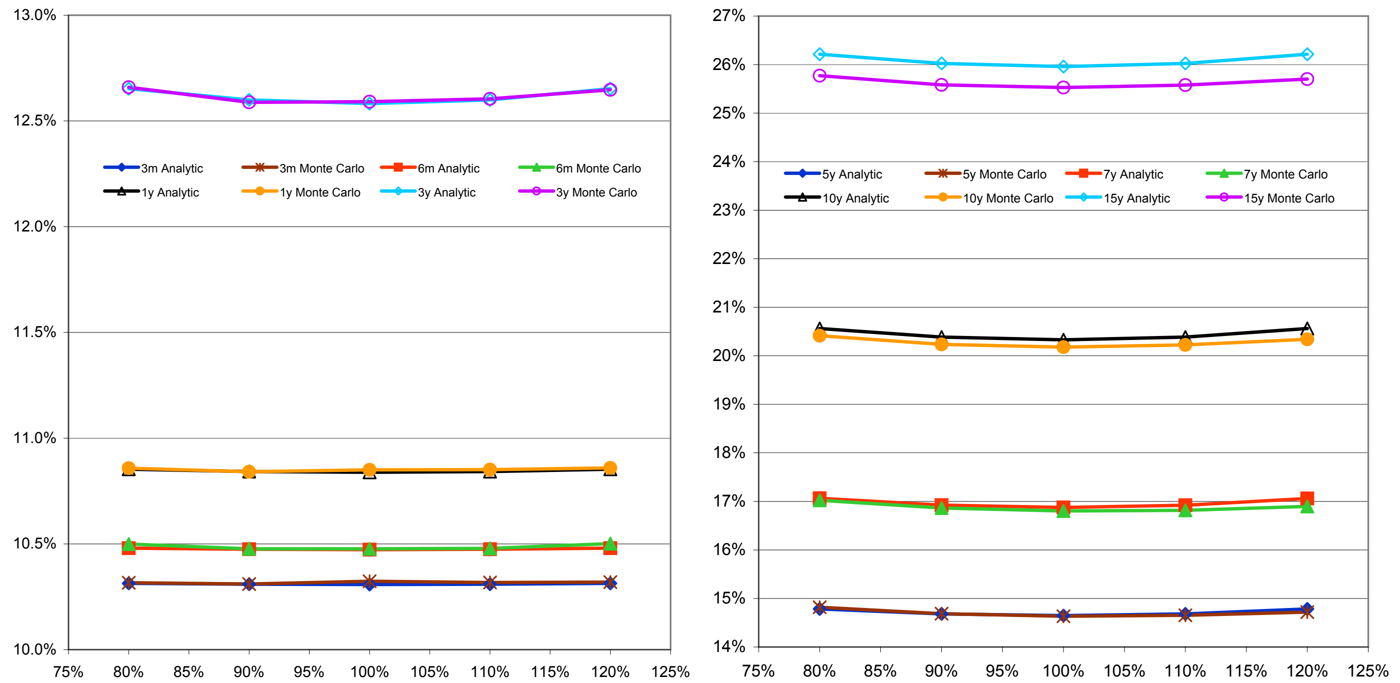

Figure 1: Numerical and analytical implied volatilities for different maturities T as a function of K/Q_T, i.e. strike divided by forward FX rate. $f_i^{\mathsf{D}} = f_i^{\mathsf{F}} = 5\%$, $s_Q = s_i^{\mathsf{D}} = s_j^{\mathsf{F}} = 0$, $\sigma_Q = 10\%$, $\sigma_i^{\mathsf{D}} = \sigma_i^{\mathsf{F}} = 40\%$, $\rho_{f_i^{\mathsf{D}} f_j^{\mathsf{D}}} = \rho_{f_i^{\mathsf{F}} f_j^{\mathsf{F}}} = 1$, $\rho_{Q,f_j^{\mathsf{D}}} = 0.3$, $\rho_{Q,f_j^{\mathsf{F}}} = -0.3$, $\rho_{f_j^{\mathsf{D}},f_j^{\mathsf{F}}} = 0.3$

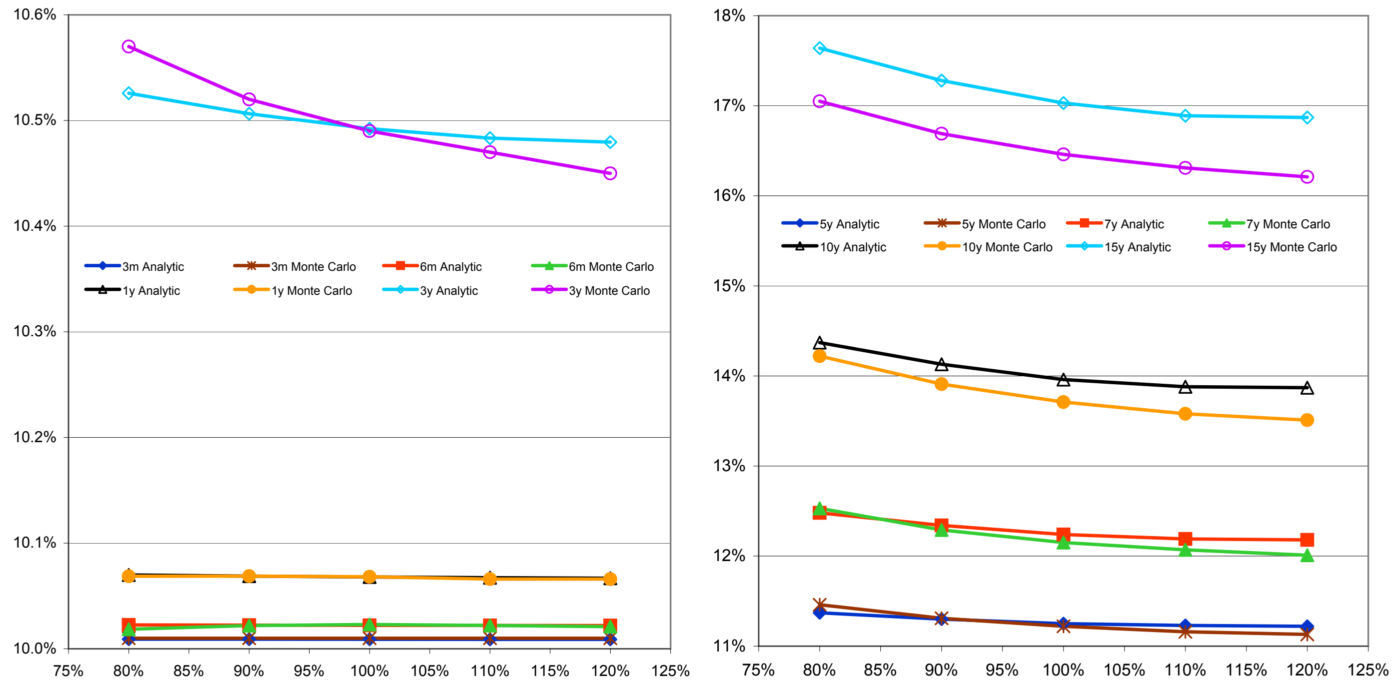

Figure 2: Numerical and analytical implied volatilities for different maturities T as a function of K/Q_T, i.e. strike divided by forward FX rate. $f_i^{\mathsf{D}} = 6\%$, $f_i^{\mathsf{F}} = 2\%$, $s_Q = s_i^{\mathsf{D}} = s_j^{\mathsf{F}} = 0$, $\sigma_Q = 10\%$, $\sigma_i^{\mathsf{D}} = 20\%$, $\sigma_i^{\mathsf{F}} = 60\%$, $\rho_{f_i^{\mathsf{D}} f_j^{\mathsf{D}}} = \rho_{f_i^{\mathsf{F}} f_j^{\mathsf{F}}} = \mathrm{e}^{-|t_i - t_j|/10}$, $\rho_{f_j^{\mathsf{D}},f_j^{\mathsf{F}}} = \rho_{Q,f_j^{\mathsf{D}}} = \rho_{Q,f_j^{\mathsf{F}}} = 0$,

different interest rate and volatility levels with fully factorised (i.e. decorrelated) interest curve dynamics. Note that the difference in interest rates and volatilities gives rise to a skew for FX implied volatilities, despite the fact that absolute interest rate volatility levels in the two currencies are approximately equal ($\sim 1.2\%$).

In figures 3 to 5, we show the analytical results in comparison to simulation data for the same overall scenario as in figure 2, but for a range of skew parameters s_Q and different maturities. The FX displaced diffusion parameter σ_Q was rescaled for different s_Q according to $\sigma_Q = 10\% \cdot Q(0)/(Q(0) + s_Q)$ which gives rise to the appearance that the implied volatility curves, with varying s_Q, pivot about the point where the FX spot is in relation to the respective FX forward. The last set of results shown in figure 6 is for a market-

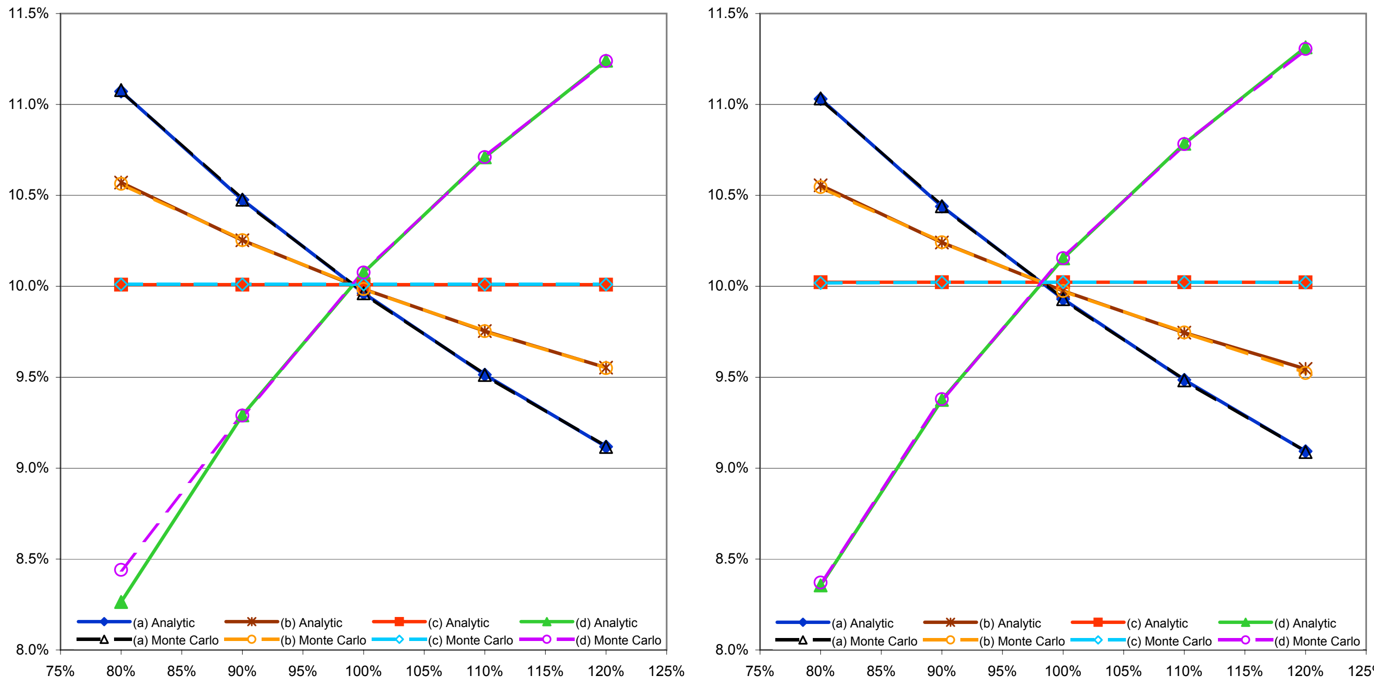

Figure 3: Numerical and analytical 3 month (left) and 6 month (right) implied volatilities with different FX skew settings: (a) $s_Q = 8\log_2(10) \cdot Q$ (almost normal), (b) $s_Q = Q$ (similar to square root distribution), (c) $s_Q = 0$ (almost lognormal), (d) $s_Q = -\log_2(3/2) \cdot Q$ (positive skew). $f_i^{\mathsf{D}} = 6\%$, $f_i^{\mathsf{F}} = 2\%$, $s_i^{\mathsf{D}} = s_j^{\mathsf{F}} = 0$, $\sigma_Q = 10\% \cdot \frac{Q}{Q+s_Q}$, $\sigma_i^{\mathsf{D}} = 20\%$, $\sigma_i^{\mathsf{F}} = 60\%$, $\rho_{f_i^{\mathsf{D}} f_j^{\mathsf{D}}} = \rho_{f_i^{\mathsf{F}} f_j^{\mathsf{F}}} = \mathrm{e}^{-|t_i - t_j|/10}$, $\rho_{f_j^{\mathsf{D}}, f_j^{\mathsf{F}}} = \rho_{Q, f_j^{\mathsf{D}}} = \rho_{Q, f_j^{\mathsf{F}}} = 0$

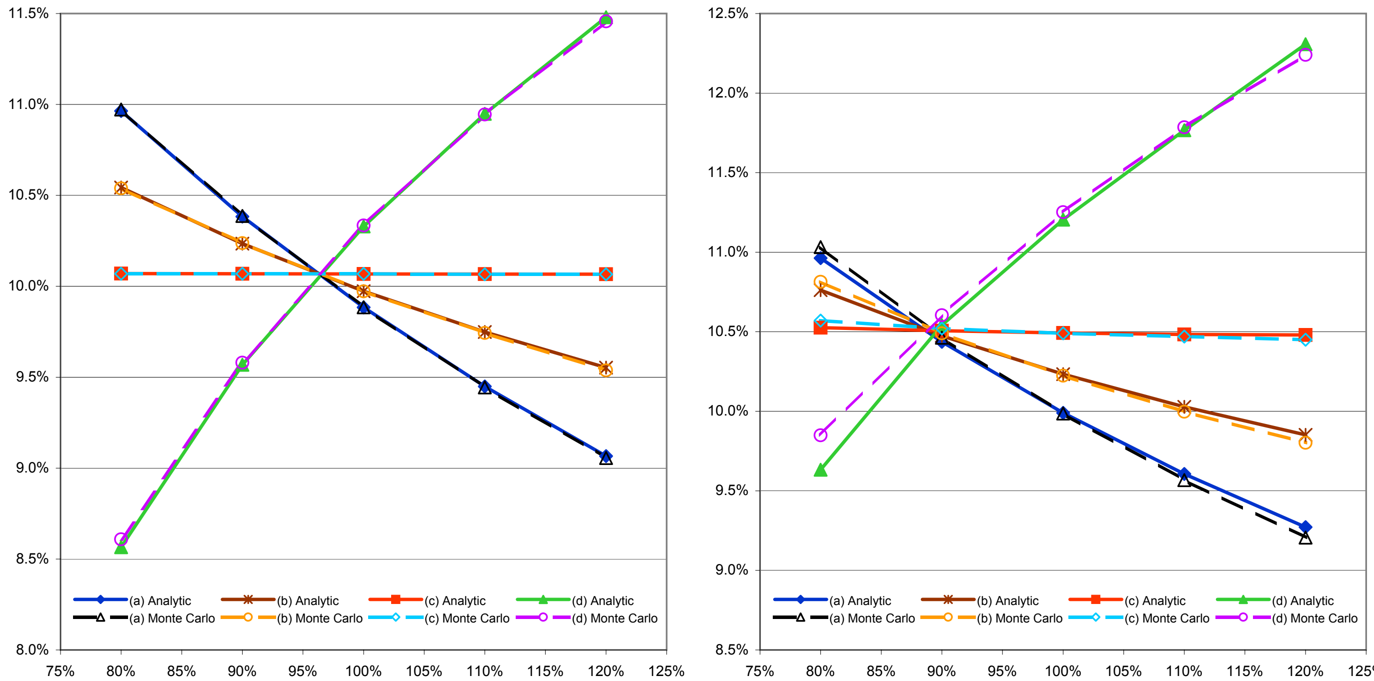

Figure 4: Numerical and analytical 1 year (left) and 3 year (right) implied volatilities with different FX skew settings: (a) $s_Q = 8\log_2(10) \cdot Q$ (almost normal), (b) $s_Q = Q$ (similar to square root distribution), (c) $s_Q = 0$ (almost lognormal), (d) $s_Q = -\log_2(3/2) \cdot Q$ (positive skew). $f_i^{\mathsf{D}} = 6\%$, $f_i^{\mathsf{F}} = 2\%$, $s_i^{\mathsf{D}} = s_j^{\mathsf{F}} = 0$, $\sigma_Q = 10\% \cdot \frac{Q}{Q+s_Q}$, $\sigma_i^{\mathsf{D}} = 20\%$, $\sigma_i^{\mathsf{F}} = 60\%$, $\rho_{f_i^{\mathsf{D}} f_j^{\mathsf{D}}} = \rho_{f_i^{\mathsf{F}} f_j^{\mathsf{F}}} = \mathrm{e}^{-|t_i - t_j|/10}$, $\rho_{f_j^{\mathsf{D}}, f_j^{\mathsf{F}}} = \rho_{Q, f_j^{\mathsf{D}}} = \rho_{Q, f_j^{\mathsf{F}}} = 0$

given USD (domestic) and EUR (foreign) interest rate scenario as seen in the market for Friday October 13, 2006. Forward rate volatility term structures were calibrated to caplet prices. Specifically, term structures of

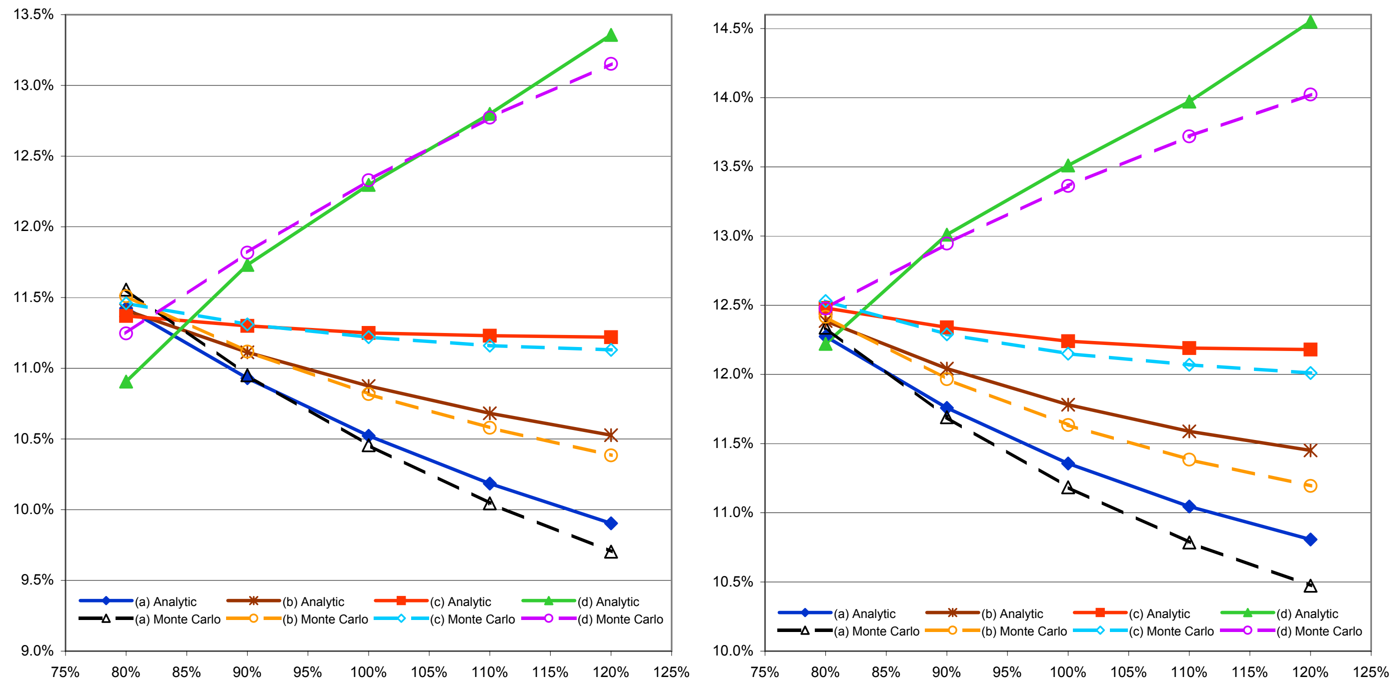

Figure 5: Numerical and analytical 5 year (left) and 7 year (right) implied volatilities with different FX skew settings: (a) $s_Q = 8\log_2(10)\cdot Q$ (almost normal), (b) $s_Q = Q$ (similar to square root distribution), (c) $s_Q = 0$ (almost lognormal), (d) $s_Q = -\log_2(3/2)\cdot Q$ (positive skew). $f_i^{\mathsf{D}} = 6\%$, $f_i^{\mathsf{F}} = 2\%$, $s_i^{\mathsf{D}} = s_j^{\mathsf{F}} = 0$, $\sigma_Q = 10\% \cdot \frac{Q}{Q+s_Q}$, $\sigma_i^{\mathsf{D}} = 20\%$, $\sigma_i^{\mathsf{F}} = 60\%$, $\rho_{f_i^{\mathsf{D}} f_j^{\mathsf{D}}} = \rho_{f_i^{\mathsf{F}} f_j^{\mathsf{F}}} = \mathrm{e}^{-|t_i - t_j|/10}$, $\rho_{f_j^{\mathsf{D}}, f_j^{\mathsf{F}}} = \rho_{Q, f_j^{\mathsf{D}}} = \rho_{Q, f_j^{\mathsf{F}}} = 0$

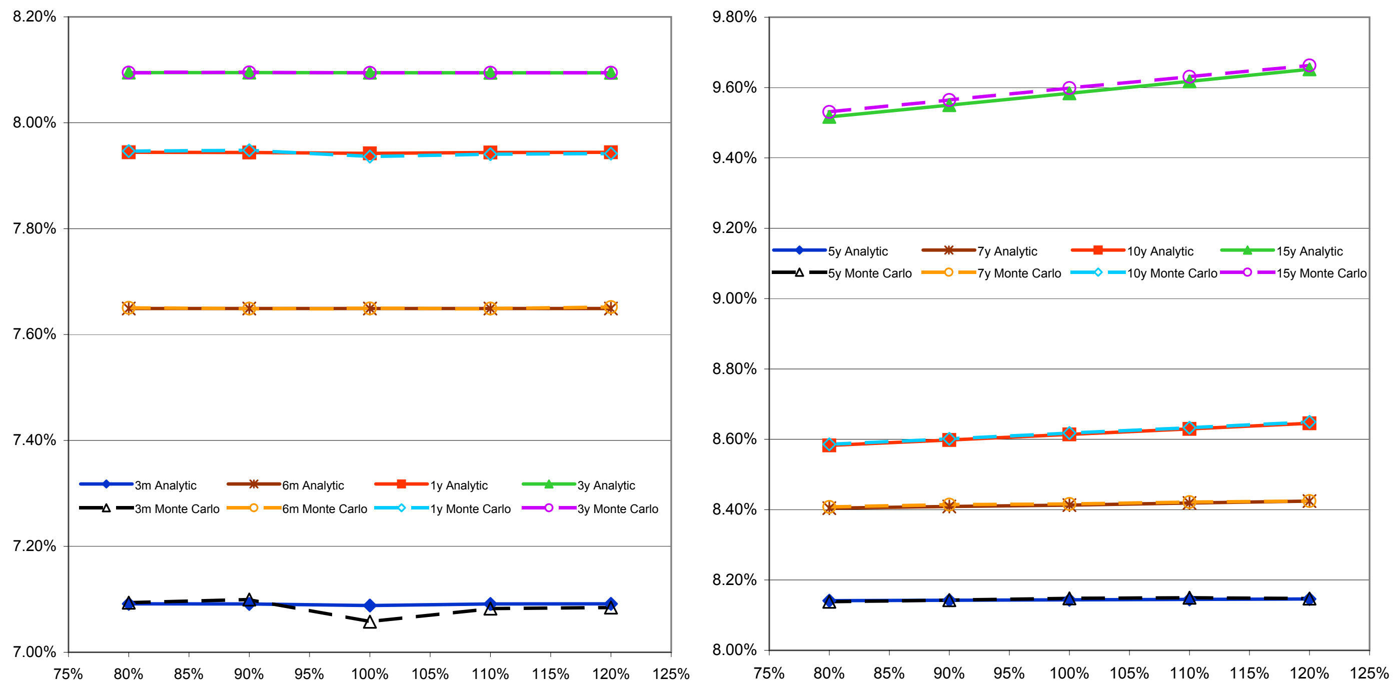

Figure 6: Numerical and analytical implied volatilities for market-calibrated USD (domestic) / EUR (foreign) rates and volatilities on Friday October 13, 2006.

instantaneous volatility of individual forward rates were defined by the parametric Nelson-Siegel form

$$\sigma(t,T) \;=\; k_T \cdot \left[(a + b \cdot (T-t)) \cdot \mathrm{e}^{-c\cdot(T-t)} + d \right] \tag{5.1}$$

with

$$\begin{array}{lllll} a = -0.074514253 & b = 0.208715347 & c = 0.606615724 & d = 0.107550229 & \text{for USD (domestic)} \\ a = -0.071209658 & b = 0.196349282 & c = 0.632737991 & d = 0.100540646 & \text{for EUR (foreign)} \end{array} \tag{5.2}$$

for the instantaneous volatility of a forward rate expiring at T. This leaves a scaling constant k_T for each forward rate expiring at T permitting calibration to market observable caplet prices. The forward rates and k_{T_i} scaling numbers are shown in figure 7. Note that the Nelson-Siegel parametrisation of instantaneous

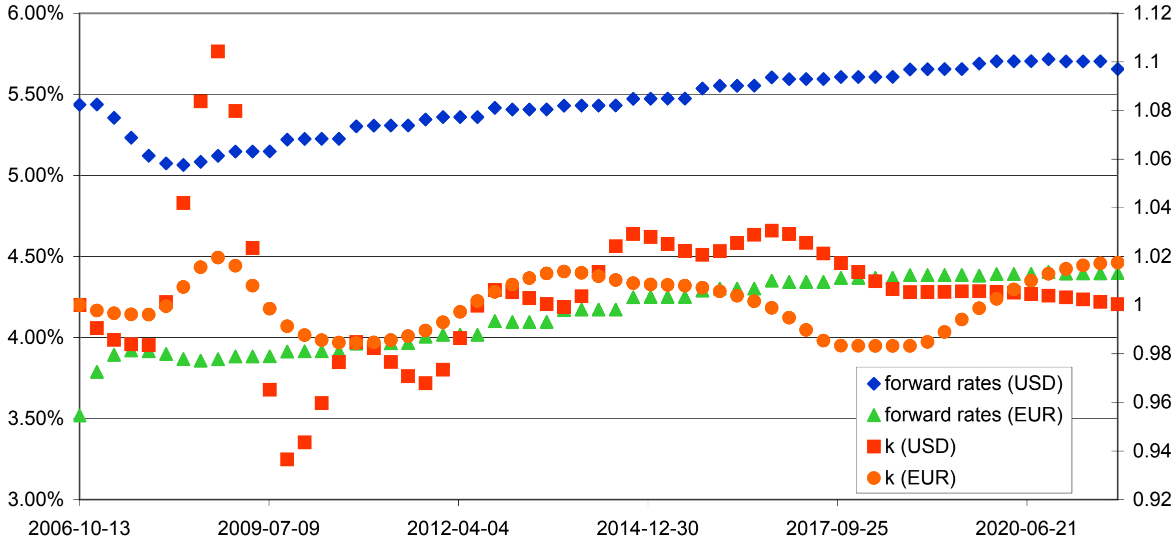

Figure 7: Forward rates (left axis) and k_{T_i} volatility scaling factors (right axis) from calibration to market on Friday October 13, 2006, as used for the results shown in figure 6.

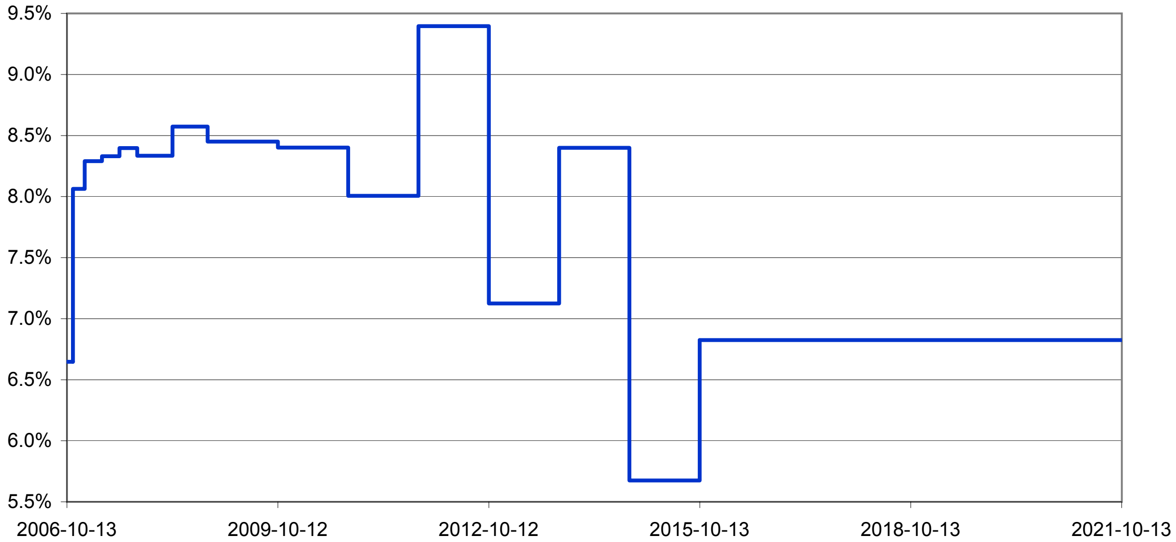

Figure 8: Instantaneous FX volatility function $\sigma_Q(t)$ calibrated to market observable plain vanilla option prices at the money on Friday October 13, 2006, as used for the results shown in figure 6.

volatility (5.1) with scaling constants k_{T_i} means that perfect time homogeneity of volatility is given when all of the k_{T_i} are identical. This is almost achieved for EUR, as can be seen in figure 7, and a reasonably high degree of time homogeneity is also given for USD since all the k_{T_i} values are near unity. Correlations between interest rates within each yield curve were given by

$$\rho_{f_i^{\mathsf{D}} f_j^{\mathsf{D}}}(t) = \rho_{f_i^{\mathsf{F}} f_j^{\mathsf{F}}}(t) = \mathrm{e}^{-\frac{1}{5}|\sqrt{t_i - t} - \sqrt{t_j - t}|} \ . \tag{5.3}$$

There was no cross-currency interest rate correlation, i.e. $\rho_{f_i^{\mathsf{D}} f_j^{\mathsf{F}}} = 0$. Correlations between domestic interest rates and the spot FX rate was $\rho_{Q, f_i^{\mathsf{D}}}(t) = -\frac{1}{4}\mathrm{e}^{-\frac{1}{5}\sqrt{t_i - t}}$ and the correlation of the FX spot rate with foreign forward rates was $\rho_{Q, f_i^{\mathsf{F}}}(t) = \frac{1}{4}\mathrm{e}^{-\frac{1}{5}\sqrt{t_i - t}}$. Displacements of forward rates were individually set to $s_i^{\mathsf{D}} = f_i^{\mathsf{D}}$ and $s_j^{\mathsf{F}} = f_j^{\mathsf{F}}$ (similar to square root distribution). The spot FX rate was undisplaced. The instantaneous FX driver volatility shown in figure 8 was calibrated as a piecewise constant function to match market observable plain vanilla option prices at the money out to ten years, and extrapolated flat beyond that. As can be seen in figure 6, for a market-realistic scenario, the proposed approximations are of superb quality even for long dated FX options.

6 Conclusion

In this article, we derived analytic approximation formulae for FX options in Libor market models. It turns out that the derived formulae are accurate enough for use in practical applications. The asymptotic expansion method was straightforwardly extended to cross currency FX markets form single currency interest markets, which was analyzed previously by Kawai [Kaw03]. The method has proved to be very powerful and flexible, whence it can be applied to other stochastic processes such as other interest rate and stochastic volatility models.

A Derivation of the FX option formula

The formula is obtained using the asymptotic expansion method. The method is fully explained in Kawai [Kaw03] by applying the method to a European swaption pricing in the LMM. To derive the formula, it is more convenient to express the stochastic differential equation (2.1), (2.2) and (2.3) as being driven by $2N+1$ independent standard Wiener processes W by decomposing the covariance structure into orthogonal components.

- The spot FX rate

$$\frac{\mathrm{d}Q}{Q} = \mu_Q\,\mathrm{d}t + \tilde{\sigma}_Q^\top\,\mathrm{d}W\ , \qquad \text{with} \qquad \mu_Q = r^{\mathsf{D}} - r^{\mathsf{F}} - \sigma_Q \sum_{k=0}^{N-1} \frac{f_k^{\mathsf{D}}\tau_k^{\mathsf{D}}}{1+f_k^{\mathsf{D}}\tau_k^{\mathsf{D}}}\sigma_k^{\mathsf{D}}\varrho_{Q,f_k^{\mathsf{D}}}\ . \tag{A.1}$$

- The domestic forward rates

$$\frac{\mathrm{d}f_i^{\mathsf{D}}}{f_i^{\mathsf{D}}} = \mu_i^{\mathsf{D}}\,\mathrm{d}t + \tilde{\sigma}_i^{\mathsf{D}\,\top}\cdot\mathrm{d}W\ , \qquad \text{with} \qquad \mu_i^{\mathsf{D}} = -\sigma_i^{\mathsf{D}} \sum_{k=i+1}^{N-1} \frac{f_k^{\mathsf{D}}\tau_k^{\mathsf{D}}}{1+f_k^{\mathsf{D}}\tau_k^{\mathsf{D}}}\sigma_k^{\mathsf{D}}\varrho_{f_i^{\mathsf{D}},f_k^{\mathsf{D}}}\ . \tag{A.2}$$

- The foreign forward rates

$$\frac{\mathrm{d}f_i^{\mathsf{F}}}{f_i^{\mathsf{F}}} = \mu_i^{\mathsf{F}}\,\mathrm{d}t + \tilde{\sigma}_i^{\mathsf{F}\,\top}\cdot\mathrm{d}W\ , \tag{A.3}$$

  with

$$\mu_i^{\mathsf{F}} = \sigma_i^{\mathsf{F}}\left(-\sigma_Q\varrho_{Q,f_i^{\mathsf{F}}} + \sum_{k=0}^{i}\frac{f_k^{\mathsf{F}}\tau_k^{\mathsf{F}}}{1+f_k^{\mathsf{F}}\tau_k^{\mathsf{F}}}\sigma_k^{\mathsf{F}}\varrho_{f_i^{\mathsf{F}},f_k^{\mathsf{F}}} - \sum_{k=0}^{N-1}\frac{f_k^{\mathsf{D}}\tau_k^{\mathsf{D}}}{1+f_k^{\mathsf{D}}\tau_k^{\mathsf{D}}}\sigma_k^{\mathsf{D}}\varrho_{f_k^{\mathsf{D}},f_k^{\mathsf{F}}}\right)\ , \tag{A.4}$$

  where $\tilde{\sigma}_Q$, $\tilde{\sigma}_i^{\mathsf{D}}$ and $\tilde{\sigma}_i^{\mathsf{F}}$ are $2N+1$ dimensional vectors satisfying

$$\tilde{\sigma}_Q^\top\cdot\tilde{\sigma}_Q = \sigma_Q^2\ , \qquad \tilde{\sigma}_Q^\top\cdot\tilde{\sigma}_i^{\mathsf{D}} = \sigma_Q\sigma_i^{\mathsf{D}}\varrho_{Q,f_i^{\mathsf{D}}}\ , \qquad \tilde{\sigma}_Q^\top\cdot\tilde{\sigma}_i^{\mathsf{F}} = \sigma_Q\sigma_i^{\mathsf{F}}\varrho_{Q,f_i^{\mathsf{F}}}\ , \tag{A.5}$$

$$\tilde{\sigma}_i^{\mathsf{D}\,\top}\cdot\tilde{\sigma}_j^{\mathsf{D}} = \sigma_i^{\mathsf{D}}\sigma_j^{\mathsf{D}}\varrho_{f_i^{\mathsf{D}},f_j^{\mathsf{D}}}\ , \qquad \tilde{\sigma}_i^{\mathsf{F}\,\top}\cdot\tilde{\sigma}_j^{\mathsf{F}} = \sigma_i^{\mathsf{F}}\sigma_j^{\mathsf{F}}\varrho_{f_i^{\mathsf{F}},f_j^{\mathsf{F}}}\ , \qquad \tilde{\sigma}_i^{\mathsf{D}\,\top}\cdot\tilde{\sigma}_j^{\mathsf{F}} = \sigma_i^{\mathsf{D}}\sigma_j^{\mathsf{F}}\varrho_{f_i^{\mathsf{D}},f_j^{\mathsf{F}}}\ . \tag{A.6}$$

From equations (A.1), (A.2) and (A.3), the T_N-forward FX rate as defined in (3.1) follows

$$\frac{\mathrm{d}Q_{T_N}(t)}{Q_{T_N}(t)} = \left(\tilde{\sigma}_Q(t)+\eta(t)\right)^\top\cdot\mathrm{d}W_t\ , \qquad \text{with} \qquad \eta = \sum_{k=0}^{N-1}\frac{f_k^{\mathsf{D}}\tau_k^{\mathsf{D}}}{1+f_k^{\mathsf{D}}\tau_k^{\mathsf{D}}}\tilde{\sigma}_k^{\mathsf{D}} - \sum_{k=0}^{N-1}\frac{f_k^{\mathsf{F}}\tau_k^{\mathsf{F}}}{1+f_k^{\mathsf{F}}\tau_k^{\mathsf{F}}}\tilde{\sigma}_k^{\mathsf{F}}\ . \tag{A.7}$$

Naturally, it is a martingale in the T_N measure which obviates any considerations regarding the domestic and foreign short rates we may have had with respect to (2.1) and similar.

Now, as is usual with asymptotic expansions, we insert a smallness parameter ε into equation (A.7) to obtain

$$Q^{(\varepsilon)}_{T_N}(T_N) = Q_{T_N}(0) + \varepsilon \int_0^{T_N} Q^{(\varepsilon)}_{T_N}(t) \left(\tilde{\sigma}_Q(t) + \eta^{(\varepsilon)}(t)\right)^{\top} \cdot \mathrm{d}W_t \,, \tag{A.8}$$

with

$$\eta^{(\varepsilon)} = \sum_{k=0}^{N-1} \frac{f_k^{\mathsf{D}\,(\varepsilon)}\,\tau_k^{\mathsf{D}}}{1+f_k^{\mathsf{D}\,(\varepsilon)}\,\tau_k^{\mathsf{D}}}\tilde{\sigma}_k^{\mathsf{D}} - \sum_{k=0}^{N-1} \frac{f_k^{\mathsf{F}\,(\varepsilon)}\,\tau_k^{\mathsf{F}}}{1+f_k^{\mathsf{F}\,(\varepsilon)}\,\tau_k^{\mathsf{F}}}\tilde{\sigma}_k^{\mathsf{F}} \,. \tag{A.9}$$

Note that the superscript (ε) is not meant to indicate the ε-th derivative but instead denotes dependence on ε. Here, perturbed interest rates follow

$$f_i^{\mathsf{D}\,(\varepsilon)} = f_i^{\mathsf{D}}(0) + \varepsilon \int_0^{T_N} f_i^{\mathsf{D}}(0)\, \mu_i^{\mathsf{D}\,(0)}\ \mathrm{d}u + \varepsilon \int_0^{T_N} f_i^{\mathsf{D}\,(\varepsilon)}\ \tilde{\sigma}_i^{\mathsf{D}\,\top} \cdot \mathrm{d}W \,, \tag{A.10}$$

with

$$\mu_i^{\mathsf{D}\,(0)} = -\sigma_i^{\mathsf{D}} \sum_{k=i+1}^{N-1} \frac{f_k^{\mathsf{D}}(0)\tau_k^{\mathsf{D}}}{1+f_k^{\mathsf{D}}(0)\tau_k^{\mathsf{D}}}\sigma_k^{\mathsf{D}}\varrho_{f_i,f_k} \,. \tag{A.11}$$

and

$$f_i^{\mathsf{F}\,(\varepsilon)} = f_i^{\mathsf{F}}(0) + \varepsilon \int_0^{T_N} f_i^{\mathsf{F}}(0)\, \mu_i^{\mathsf{F}\,(0)}\ \mathrm{d}u + \varepsilon \int_0^{T_N} f_i^{\mathsf{F}\,(\varepsilon)}\ \tilde{\sigma}_i^{\mathsf{F}\,\top} \cdot \mathrm{d}W \,, \tag{A.12}$$

as well as

$$\mu_i^{\mathsf{F}\,(0)} = \sigma_i^{\mathsf{F}} \left(\sum_{k=0}^{i} \frac{f_k^{\mathsf{F}}(0)\tau_k^{\mathsf{F}}}{1+f_k^{\mathsf{F}}(0)\tau_k^{\mathsf{F}}}\sigma_k^{\mathsf{F}}\varrho_{f_i^{\mathsf{F}},f_k^{\mathsf{F}}} - \sigma_Q\varrho_{f_i^{\mathsf{F}},Q} - \sum_{k=0}^{N-1} \frac{f_k^{\mathsf{D}}(0)\tau_k^{\mathsf{D}}}{1+f_k^{\mathsf{D}}(0)\tau_k^{\mathsf{D}}}\sigma_k^{\mathsf{D}}\varrho_{f_i^{\mathsf{F}},f_k} \right) \,. \tag{A.13}$$

Notice that the drift terms of the interest rates are approximated deterministically using initial interest rates as a consequence of the Itô-Taylor expansions in (A.8), (A.10), and (A.12). This makes both the derivation and the resulting formula simpler, yet it remains accurate. By applying a Taylor series expansion in ε to the forward FX rate (A.8), we obtain a third-order asymptotic expansion.

$$Q^{(\varepsilon)}_{T_N}(T_N) = Q_{T_N}(0) + \varepsilon\, \frac{\partial Q^{(\varepsilon)}}{\partial \varepsilon}\bigg|_{\varepsilon=0} + \frac{1}{2}\varepsilon^2\, \frac{\partial^2 Q^{(\varepsilon)}}{\partial \varepsilon^2}\bigg|_{\varepsilon=0} + \frac{1}{6}\varepsilon^3\, \frac{\partial^3 Q^{(\varepsilon)}}{\partial \varepsilon^3}\bigg|_{\varepsilon=0} + \mathcal{O}\left(\varepsilon^4\right) \,, \tag{A.14}$$

where

$$\frac{\partial Q^{(\varepsilon)}}{\partial \varepsilon}\bigg|_{\varepsilon=0} = Q_{T_N}(0) \int_0^{T_N} \left(\tilde{\sigma}_Q + \eta^{(0)}\right)^{\top} \cdot \mathrm{d}W_u \,, \tag{A.15}$$

$$\begin{aligned}
\frac{\partial^2 Q^{(\varepsilon)}}{\partial \varepsilon^2}\bigg|_{\varepsilon=0} = {} & 2Q_{T_N}(0) \int_0^{T_N}\int_0^{u_1} \left(\tilde{\sigma}_Q + \eta^{(0)}\right)^{\top} \cdot \mathrm{d}W_{u_2} \left(\tilde{\sigma}_Q + \eta^{(0)}\right)^{\top} \cdot \mathrm{d}W_{u_1} \\
& + 2Q_{T_N}(0) \sum_{k=0}^{N-1} \frac{f_k^{\mathsf{D}}(0)\tau_k^{\mathsf{D}}}{\left(1+f_k^{\mathsf{D}}(0)\tau_k^{\mathsf{D}}\right)^2} \left(\int_0^{T_N}\int_0^{u_1} \mu_k^{\mathsf{D}\,(0)}\ \mathrm{d}u_2\ \tilde{\sigma}_k^{\mathsf{D}\,\top} \cdot \mathrm{d}W_{u_1} + \int_0^{T_N}\int_0^{u_1} \tilde{\sigma}_k^{\mathsf{D}\,\top} \cdot \mathrm{d}W_{u_2}\ \tilde{\sigma}_k^{\mathsf{D}\,\top} \cdot \mathrm{d}W_{u_1} \right) \\
& - 2Q_{T_N}(0) \sum_{k=0}^{N-1} \frac{f_k^{\mathsf{F}}(0)\tau_k^{\mathsf{F}}}{\left(1+f_k^{\mathsf{F}}(0)\tau_k^{\mathsf{F}}\right)^2} \left(\int_0^{T_N}\int_0^{u_1} \mu_k^{\mathsf{F}\,(0)}\ \mathrm{d}u_2\ \tilde{\sigma}_k^{\mathsf{F}\,\top} \cdot \mathrm{d}W_{u_1} + \int_0^{T_N}\int_0^{u_1} \tilde{\sigma}_k^{\mathsf{F}\,\top} \cdot \mathrm{d}W_{u_2}\ \tilde{\sigma}_k^{\mathsf{F}\,\top} \cdot \mathrm{d}W_{u_1} \right) .
\end{aligned} \tag{A.16}$$

and

$$
\begin{aligned}
\left.\frac{\partial^3 Q^{(\varepsilon)}}{\partial \varepsilon^3}\right|_{\varepsilon=0} = \quad & 6Q_{T_N}(0) \int_0^{T_N}\int_0^{u_1}\int_0^{u_2} \left(\tilde{\sigma}_Q + \eta^{(0)}\right)^{\top} \cdot \mathrm{d}W_{u_3} \left(\tilde{\sigma}_Q + \eta^{(0)}\right)^{\top} \cdot \mathrm{d}W_{u_2} \left(\tilde{\sigma}_Q + \eta^{(0)}\right)^{\top} \cdot \mathrm{d}W_{u_1} \\
& + 6Q_{T_N}(0) \sum_{k=0}^{N-1} \frac{f_k^{\mathsf{D}}(0)\tau_k^{\mathsf{D}}}{\left(1+f_k^{\mathsf{D}}(0)\tau_k^{\mathsf{D}}\right)^2} \int_0^{T_N}\int_0^{u_1}\int_0^{u_2} \tilde{\sigma}_k^{\mathsf{D}\,\top} \cdot \mathrm{d}W_{u_3}\, \tilde{\sigma}_k^{\mathsf{D}\,\top} \cdot \mathrm{d}W_{u_2} \left(\tilde{\sigma}_Q + \eta^{(0)}\right)^{\top} \cdot \mathrm{d}W_{u_1} \\
& - 6Q_{T_N}(0) \sum_{k=0}^{N-1} \frac{f_k^{\mathsf{F}}(0)\tau_k^{\mathsf{F}}}{\left(1+f_k^{\mathsf{F}}(0)\tau_k^{\mathsf{F}}\right)^2} \int_0^{T_N}\int_0^{u_1}\int_0^{u_2} \tilde{\sigma}_k^{\mathsf{F}\,\top} \cdot \mathrm{d}W_{u_3}\, \tilde{\sigma}_k^{\mathsf{F}\,\top} \cdot \mathrm{d}W_{u_2} \left(\tilde{\sigma}_Q + \eta^{(0)}\right)^{\top} \cdot \mathrm{d}W_{u_1} \\
& + 6Q_{T_N}(0) \sum_{k=0}^{N-1} \frac{f_k^{\mathsf{D}}(0)\tau_k^{\mathsf{D}}}{\left(1+f_k^{\mathsf{D}}(0)\tau_k^{\mathsf{D}}\right)^2} \int_0^{T_N}\int_0^{u_1}\int_0^{u_2} \tilde{\sigma}_k^{\mathsf{D}\,\top} \cdot \mathrm{d}W_{u_3} \left(\tilde{\sigma}_Q + \eta^{(0)}\right)^{\top} \cdot \mathrm{d}W_{u_2}\, \tilde{\sigma}_k^{\mathsf{D}\,\top} \cdot \mathrm{d}W_{u_1} \\
& - 6Q_{T_N}(0) \sum_{k=0}^{N-1} \frac{f_k^{\mathsf{F}}(0)\tau_k^{\mathsf{F}}}{\left(1+f_k^{\mathsf{F}}(0)\tau_k^{\mathsf{F}}\right)^2} \int_0^{T_N}\int_0^{u_1}\int_0^{u_2} \tilde{\sigma}_k^{\mathsf{F}\,\top} \cdot \mathrm{d}W_{u_3} \left(\tilde{\sigma}_Q + \eta^{(0)}\right)^{\top} \cdot \mathrm{d}W_{u_2}\, \tilde{\sigma}_k^{\mathsf{F}\,\top} \cdot \mathrm{d}W_{u_1} \\
& + 6Q_{T_N}(0) \sum_{k=0}^{N-1} \frac{f_k^{\mathsf{D}}(0)\tau_k^{\mathsf{D}}}{\left(1+f_k^{\mathsf{D}}(0)\tau_k^{\mathsf{D}}\right)^2} \int_0^{T_N}\int_0^{u_1}\int_0^{u_2} \left(\tilde{\sigma}_Q + \eta^{(0)}\right)^{\top} \cdot \mathrm{d}W_{u_3}\, \tilde{\sigma}_k^{\mathsf{D}\,\top} \cdot \mathrm{d}W_{u_2}\, \tilde{\sigma}_k^{\mathsf{D}\,\top} \cdot \mathrm{d}W_{u_1} \\
& - 6Q_{T_N}(0) \sum_{k=0}^{N-1} \frac{f_k^{\mathsf{F}}(0)\tau_k^{\mathsf{F}}}{\left(1+f_k^{\mathsf{F}}(0)\tau_k^{\mathsf{F}}\right)^2} \int_0^{T_N}\int_0^{u_1}\int_0^{u_2} \left(\tilde{\sigma}_Q + \eta^{(0)}\right)^{\top} \cdot \mathrm{d}W_{u_3}\, \tilde{\sigma}_k^{\mathsf{F}\,\top} \cdot \mathrm{d}W_{u_2}\, \tilde{\sigma}_k^{\mathsf{F}\,\top} \cdot \mathrm{d}W_{u_1} \\
& + 6Q_{T_N}(0) \sum_{k=0}^{N-1} \frac{f_k^{\mathsf{D}}(0)\tau_k^{\mathsf{D}}}{\left(1+f_k^{\mathsf{D}}(0)\tau_k^{\mathsf{D}}\right)^2} \int_0^{T_N}\int_0^{u_1} \tilde{\sigma}_k^{\mathsf{D}\,\top} \cdot \left(\tilde{\sigma}_Q + \eta^{(0)}\right) \mathrm{d}u_2\, \tilde{\sigma}_k^{\mathsf{D}\,\top} \cdot \mathrm{d}W_{u_1} \\
& - 6Q_{T_N}(0) \sum_{k=0}^{N-1} \frac{f_k^{\mathsf{F}}(0)\tau_k^{\mathsf{F}}}{\left(1+f_k^{\mathsf{F}}(0)\tau_k^{\mathsf{F}}\right)^2} \int_0^{T_N}\int_0^{u_1} \tilde{\sigma}_k^{\mathsf{F}\,\top} \cdot \left(\tilde{\sigma}_Q + \eta^{(0)}\right) \mathrm{d}u_2\, \tilde{\sigma}_k^{\mathsf{F}\,\top} \cdot \mathrm{d}W_{u_1} \\
& + 6Q_{T_N}(0) \sum_{k=0}^{N-1} \frac{f_k^{\mathsf{D}}(0)\tau_k^{\mathsf{D}}}{\left(1+f_k^{\mathsf{D}}(0)\tau_k^{\mathsf{D}}\right)^2} \int_0^{T_N}\int_0^{u_1}\int_0^{u_2} \tilde{\sigma}_k^{\mathsf{D}\,\top} \cdot \mathrm{d}W_{u_3}\, \tilde{\sigma}_k^{\mathsf{D}\,\top} \cdot \mathrm{d}W_{u_2}\, \tilde{\sigma}_k^{\mathsf{D}\,\top} \cdot \mathrm{d}W_{u_1} \\
& - 6Q_{T_N}(0) \sum_{k=0}^{N-1} \frac{f_k^{\mathsf{F}}(0)\tau_k^{\mathsf{F}}}{\left(1+f_k^{\mathsf{F}}(0)\tau_k^{\mathsf{F}}\right)^2} \int_0^{T_N}\int_0^{u_1}\int_0^{u_2} \tilde{\sigma}_k^{\mathsf{F}\,\top} \cdot \mathrm{d}W_{u_3}\, \tilde{\sigma}_k^{\mathsf{F}\,\top} \cdot \mathrm{d}W_{u_2}\, \tilde{\sigma}_k^{\mathsf{F}\,\top} \cdot \mathrm{d}W_{u_1} + \mathcal{O}\left((f\tau)^2\right) \ . \qquad \text{(A.17)}
\end{aligned}
$$

Note that in the above expressions many dependencies on u_3, u_2, or u_1 are not mentioned explicitly for the sake of legibility.

Next, define $X^{(\varepsilon)}$ as

$$
X^{(\varepsilon)} = \frac{1}{\varepsilon}\left(Q_{T_N}^{(\varepsilon)}(T_N) - Q_{T_N}(0)\right) \ . \qquad \text{(A.18)}
$$

Then we can find an asymptotic expansion for the density function of $X^{(\varepsilon)}$ as

$$
\begin{aligned}
f_X^{(\varepsilon)}(x) &= \varphi_{v_0}(x) + \varepsilon\frac{c_1}{v_0}\left(x^3 - 3v_0 x\right)\varphi_{v_0}(x) + \varepsilon c_4\left(x^2 - v_0\right)\varphi_{v_0}(x) \\
&\quad + \varepsilon^2\frac{g_1+g_3}{v_0}\left(x^4 - 6v_0x^2 + 3v_0^2\right)\varphi_{v_0}(x) + \varepsilon^2 g_2\left(x^2 - v_0\right)\varphi_{v_0}(x) \\
&\quad + \frac{1}{2}\varepsilon^2\frac{c_1^2}{v_0^2}\left(x^6 - 15v_0x^4 + 45v_0^2x^2 - 15v_0^3\right)\varphi_{v_0}(x) + \frac{1}{2}\varepsilon^2\frac{d_1}{v_0}\left(x^4 - 6v_0x^2 + 3v_0^2\right)\varphi_{v_0}(x)
\end{aligned}
$$

$$+\frac{1}{2}\varepsilon^2 d_2\left(x^2-v_0\right)\varphi_{v_0}(x)+\mathcal{O}\left(\varepsilon^3,x\right)\ , \tag{A.19}$$

where $\varphi_{v_0}(x)$ is the Gaussian density function with mean 0 and variance

$$v_0 \ = \ Q_{T_N}^2 v_1 \tag{A.20}$$

and v_1 defined in (3.10), and c_1, c_2, c_3 and c_4 are constants defined in (3.13), (3.14), (3.15) and (3.16) respectively. As a result, letting $\varepsilon=1$, it follows that an asymptotic FX call option price is

$$P_{T_N}^{\mathsf{D}}\cdot\left[G\left(v_0\right)-2c_1v_0g_0G'\left(v_0\right)+2c_1^2v_0^2g_0^2G''\left(v_0\right)+v_0\left(c_2g_0^2+c_3v_0+2c_3v_0\right)G'\left(v_0\right)\right]\ , \tag{A.21}$$

where the function $G\left(x\right)$ is defined as

$$G\left(x\right)=g_0\Phi\left(\frac{g_0}{\sqrt{x}}\right)+x\frac{e^{-g_0^2/2x}}{\sqrt{2\pi x}}\ ,\qquad\text{with}\qquad g_0=Q_{T_N}-K\ . \tag{A.22}$$

Since the dynamics of the forward FX rate is close to lognormal, a further procedure[2] that matches coefficients to an analogous expansion for the standard Black model within the same order in ε improves the accuracy. Finally, we obtain the FX option formulae (3.2) and (3.3).

B Derivation of the FX option formula with skew

From the SDE (4.1), (4.2) and (4.3), the forward FX rate follows

$$\frac{\mathrm{d}Q_{T_N}}{Q_{T_N}}=\left(\left(1+\frac{s_Q}{Q}\right)\tilde{\sigma_Q}+\eta\right)^{\top}\cdot\mathrm{d}W\ ,\quad\text{with}\quad\eta=\sum_{k=0}^{N-1}\frac{(f_k^{\mathsf{D}}+s_k^{\mathsf{D}})\tau_k^{\mathsf{D}}}{1+f_k^{\mathsf{D}}\tau_k^{\mathsf{D}}}\tilde{\sigma}_k^{\mathsf{D}}-\sum_{k=0}^{N-1}\frac{(f_k^{\mathsf{F}}+s_k^{\mathsf{F}})\tau_k^{\mathsf{F}}}{1+f_k^{\mathsf{F}}\tau_k^{\mathsf{F}}}\tilde{\sigma}_k^{\mathsf{F}}\ . \tag{B.1}$$

Now, by allowing a small perturbation ε, we can rewrite the SDE (B.1) as

$$Q_{T_N}^{(\varepsilon)}(T_N)=Q_{T_N}(0)+\varepsilon\int_0^{T_N}Q_{T_N}^{(\varepsilon)}\left(\left(1+\frac{s_Q}{Q}\right)\tilde{\sigma}_Q+\eta^{(\varepsilon)}\right)^{\top}\cdot\mathrm{d}W\ , \tag{B.2}$$

with

$$\eta^{(\varepsilon)}=\sum_{k=0}^{N-1}\frac{\left(f_k^{\mathsf{D}\,(\varepsilon)}+s_k^{\mathsf{D}}\right)\tau_k^{\mathsf{D}}}{1+f_k^{\mathsf{D}\,(\varepsilon)}\tau_k^{\mathsf{D}}}\tilde{\sigma}_k^{\mathsf{D}}-\sum_{k=0}^{N-1}\frac{\left(f_k^{\mathsf{F}\,(\varepsilon)}+s_k^{\mathsf{F}}\right)\tau_k^{\mathsf{F}}}{1+f_k^{\mathsf{F}\,(\varepsilon)}\tau_k^{\mathsf{F}}}\tilde{\sigma}_k^{\mathsf{F}}\ . \tag{B.3}$$

The perturbed spot FX rate follows

$$Q^{(\varepsilon)}=Q(0)+\int_0^{T_N}\mu_Q^{(\varepsilon)}\,\mathrm{d}u+\varepsilon\int_0^{T_N}\left(\tilde{\sigma}_Q\right)^{\top}\mathrm{d}W\ , \tag{B.4}$$

with

$$\mu_Q^{(\varepsilon)}=Q\left(r^{\mathsf{D}}-r^{\mathsf{F}}\right)-\varepsilon\sigma_Q\sum_{k=0}^{N-1}\frac{\left(f_k^{\mathsf{D}}(0)+s_k^{\mathsf{D}}\right)\tau_k^{\mathsf{D}}}{1+f_k^{\mathsf{D}}(0)\tau_k^{\mathsf{D}}}\sigma_k^{\mathsf{D}}\varrho_{Q,f_k^{\mathsf{D}}}\ , \tag{B.5}$$

and

$$Q^{(0)}=Q(0)\frac{P_{T_N}^{\mathsf{D}}(0)}{P_{T_N}^{\mathsf{F}}(0)}\ . \tag{B.6}$$

---

[2] whose details are explained in [Kaw03]

Here, perturbed interest rates follow

$$f_i^{\mathsf{D}\,(\varepsilon)} = f_i^{\mathsf{D}}(0) + \varepsilon \int_0^{T_N} \left(f_i^{\mathsf{D}}(0) + s_i^{\mathsf{D}}\right) \mu_i^{\mathsf{D}\,(0)}\, \mathrm{d}u + \varepsilon \int_0^{T_N} \left(f_i^{\mathsf{D}\,(\varepsilon)} + s_i^{\mathsf{D}}\right) \left(\tilde{\sigma}_i^{\mathsf{D}}\right)^\top \mathrm{d}W\,, \tag{B.7}$$

with

$$\mu_i^{\mathsf{D}\,(0)} = -\sigma_i^{\mathsf{D}} \sum_{k=i+1}^{N-1} \frac{\left(f_k^{\mathsf{D}}(0) + s_k^{\mathsf{D}}\right)\tau_k^{\mathsf{D}}}{1 + f_k^{\mathsf{D}}(0)\tau_k^{\mathsf{D}}} \sigma_k^{\mathsf{D}} \varrho_{f_i, f_k}\ . \tag{B.8}$$

and

$$f_i^{\mathsf{F}\,(\varepsilon)} = f_i^{\mathsf{F}}(0) + \varepsilon \int_0^{T_N} \left(f_i^{\mathsf{F}}(0) + s_i^{\mathsf{F}}\right) \mu_i^{\mathsf{F}\,(0)}\, \mathrm{d}u + \varepsilon \int_0^{T_N} \left(f_i^{\mathsf{F}\,(\varepsilon)} + s_i^{\mathsf{F}}\right) \left(\tilde{\sigma}_i^{\mathsf{F}}\right)^\top \mathrm{d}W\,, \tag{B.9}$$

as well as

$$\mu_i^{\mathsf{F}\,(0)} = \sigma_i^{\mathsf{F}} \left(\sum_{k=0}^{i} \frac{\left(f_k^{\mathsf{F}}(0) + s_k^{\mathsf{F}}\right)\tau_k^{\mathsf{F}}}{1 + f_k^{\mathsf{F}}(0)\tau_k^{\mathsf{F}}} \sigma_k^{\mathsf{F}} \varrho_{f_i^{\mathsf{F}}, f_k^{\mathsf{F}}} - \sigma_Q \varrho_{f_i^{\mathsf{F}}, Q} - \sum_{k=0}^{N-1} \frac{\left(f_k^{\mathsf{D}}(0) + s_k^{\mathsf{D}}\right)\tau_k^{\mathsf{D}}}{1 + f_k^{\mathsf{D}}(0)\tau_k^{\mathsf{D}}} \sigma_k^{\mathsf{D}} \varrho_{f_i^{\mathsf{F}}, f_k} \right)\ . \tag{B.10}$$

Applying an asymptotic method in this setting, we obtain the FX option formula (4.5) and (4.6).

References

[AM06a] A. Antonov and T. Misirpashaev. Efficient Calibration to FX Options by Markovian Projection in Cross-Currency LIBOR Market Models. Technical report, Numerix, October 2006. ssrn.com/abstract=936087.

[AM06b] A. Antonov and T. Misirpashaev. Markovian Projection onto a Displaced Diffusion: Generic Formulas with Applications. Technical report, Numerix, October 2006. ssrn.com/abstract=937860.

[BGM97] A. Brace, D. Gatarek, and M. Musiela. The Market Model of Interest Rate Dynamics. *Mathematical Finance*, 7(2):127–155, 1997.

[HW00] J. Hull and A. White. Forward rate volatilities, swap rate volatilities, and the implementation of the LIBOR market model. *Journal of Fixed Income*, 10(2):46–62, September 2000.

[Jam97] F. Jamshidian. Libor and swap market models and measures. *Finance and Stochastics*, 1(4):293–330, 1997.

[JR00] P. Jäckel and R. Rebonato. The link between caplet and swaption volatilities in a Brace-Gatarek-Musiela/Jamshidian framework: approximate solutions and empirical evidence. *Journal of Computational Finance*, 6(4):41–59, 2003 (submitted in 2000). www.jaeckel.org/LinkingCapletAndSwaptionVolatilities.pdf.

[Kaw02] A. Kawai. Analytical and Monte Carlo swaption pricing under the forward swap measure. *Journal of Computational Finance*, 6(1):101–111, 2002. www.maths.unsw.edu.au/statistics/files/preprint-2002-12.pdf.

[Kaw03] A. Kawai. A new approximate swaption formula in the LIBOR market model: an asymptotic expansion approach. *Applied Mathematical Finance*, 10(1):49–74, March 2003. www.maths.unsw.edu.au/statistics/files/preprint-2002-13.pdf.

[KT01] N. Kunitomo and A. Takahashi. The asymptotic expansion approach to the valuation of interest rate contingent claims. *Mathematical Finance*, 11(1):117–151, January 2001.

[KT03] N. Kunitomo and A. Takahashi. Applications of the Asymptotic Expansion Approach based on Malliavin-Watanabe Calculus in Financial Problems. CIRJE F-Series CIRJE-F-245, CIRJE, Faculty of Economics, University of Tokyo, November 2003. www.e.u-tokyo.ac.jp/cirje/research/dp/2003/2003cf245.pdf.

[MSS97] K. R. Miltersen, K. Sandmann, and D. Sondermann. Closed-form solutions for term structure derivatives with lognormal interest rates. *Journal of Finance*, 52(1):409–430, 1997.

[Osa06] Y. Osajima. The asymptotic expansion formula of implied volatility for dynamic SABR model and FX hybrid model. Technical report, University of Tokyo, Graduate School of Mathematical Sciences, Komaba, Tokyo, Japan, 2006. kyokan.ms.u-tokyo.ac.jp/users/preprint/pdf/2006-29.pdf.

[Rub83] M. Rubinstein. Displaced diffusion option pricing. *Journal of Finance*, 38:213–217, March 1983.

[Sch02] E. Schlögl. A multicurrency extension of the lognormal interest rate market models. *Finance and Stochastics*, 6:173–196, 2002.

[Wat87] S. Watanabe. Analysis of Wiener functionals (Malliavin calculus) and its application to heat kernels. *The Annals of Probability*, 15:1–39, 1987.