Implied Normal Volatility

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Abstract

We give an analytical formula for the calculation of implied normal volatility (also known as *Bachelier* volatility) from vanilla option prices.

Introduction

The Bachelier vanilla option price formula is given by

$$v \;=\; \breve{\sigma}\cdot\sqrt{T}\cdot\varphi\left(\tfrac{F-K}{\breve{\sigma}\cdot\sqrt{T}}\right)+\theta\cdot(F-K)\cdot\Phi\left(\theta\cdot\tfrac{F-K}{\breve{\sigma}\cdot\sqrt{T}}\right)\,, \tag{1.1}$$

with $\theta=\pm1$ for calls/puts, and $\breve{\sigma}$ is the *normal* or *Bachelier* implied volatility. We define

$$\tilde{\Phi}(x) \;:=\; \Phi(x)+\frac{\varphi(x)}{x} \tag{1.2}$$

and

$$x \;:=\; -\theta\cdot\frac{(K-F)}{\breve{\sigma}\cdot\sqrt{T}} \tag{1.3}$$

and solve the equation

$$\tilde{\Phi}(x) \;=\; \underbrace{-\theta\cdot\frac{v}{(K-F)}}_{known} \tag{1.4}$$

whose right hand side is negative for out of the money options, where $x<0$. Finally, we obtain the Bachelier volatility $\breve{\sigma}$ as

$$\breve{\sigma} \;:=\; -\theta\cdot\left(\frac{K-F}{x\cdot\sqrt{T}}\right)\,. \tag{1.5}$$

An interesting representation of $\tilde{\Phi}(\cdot)$ as a special case of the inverse incomplete Gamma function exists [RR09], and this has led in [Gru11] to the comment "*that there are efficient algorithms to compute the inverse of the incomplete Gamma function. In particular, it is implemented in Matlab. Therefore, it is always easy to get the implied normal volatility from call*

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prices". Alas, in a commercial environment it is of little significance what may be *implemented in Matlab*, or indeed elsewhere, unless the algorithm can be extracted and ported to the respectively relevant industrial system or analytics library. In addition, all published algorithms for the inversion of the general (incomplete) Gamma function involve iterative root-finding, and are even more complex than a simple root-search for the solution of $\tilde{\Phi}(x) = y$ due to the underlying objective function being for the *generic* (incomplete) Gamma function. What's more, a concise and fully analytic implementation that is always accurate to machine precision is obviously preferable.

Remark 1.1. Unlike Black's volatility,

implied Bachelier volatility calculation is a univariate problem.

Remark 1.2. In order to condition the internal calculation on out-of-the money options, in practice, we always set

$$\tilde{\Phi}^* \;:=\; -\frac{\left|v - (\theta \cdot (F-K))_+\right|}{|K-F|}\;, \tag{1.6}$$

solve

$$\tilde{\Phi}(x^*) \;=\; \tilde{\Phi}^*\;, \tag{1.7}$$

for $x^* < 0$, and set

$$\check{\sigma} \;:=\; \frac{|K-F|}{|x \cdot \sqrt{T}|}\;. \tag{1.8}$$

Inverting the univariate objective function $\tilde{\Phi}(\cdot)$

We assume $\tilde{\Phi}^* < 0$ by conditioning on out-of-the money options and solve $\tilde{\Phi}(x^*) = \tilde{\Phi}^*$.

If $\tilde{\Phi}^* < \tilde{\Phi}^*_{\mathrm{C}}$ with $\tilde{\Phi}^*_{\mathrm{C}} := \tilde{\Phi}(-9/4) \approx -0.001882039271$, set

$$g \;:=\; 1/(\tilde{\Phi}^* - {}^1\!/_2) \tag{2.1}$$

$$\bar{\xi} \;:=\; \frac{0.032114372355 - g^2 \cdot (0.016969777977 - g^2 \cdot (2.6207332461\text{E}{-3} - 9.6066952861\text{E}{-5} \cdot g^2))}{1 - g^2 \cdot (0.6635646938 - g^2 \cdot (0.14528712196 - 0.010472855461 \cdot g^2))} \tag{2.2}$$

$$\bar{x} \;:=\; g \cdot \left(\tfrac{1}{\sqrt{2\pi}} + \bar{\xi} \cdot g^2\right) \tag{2.3}$$

else set

$$h \;:=\; \sqrt{-\ln(-\tilde{\Phi}^*)} \tag{2.4}$$

$$\bar{x} \;:=\; \frac{9.4883409779 - h \cdot (9.6320903635 - h \cdot (0.58556997323 + 2.1464093351 \cdot h))}{1 - h \cdot (0.65174820867 + h \cdot (1.5120247828 + 6.6437847132\text{E}{-5} \cdot h))} \tag{2.5}$$

Then,

$$x^* \;:=\; \bar{x} + \frac{3q\bar{x}^2 \cdot (2 - q\bar{x} \cdot (2 + \bar{x}^2))}{6 + q\bar{x} \cdot (-12 + \bar{x} \cdot (6q + \bar{x} \cdot (-6 + q\bar{x} \cdot (3 + \bar{x}^2))))} \tag{2.6}$$

with

$$q \;:=\; \frac{\tilde{\Phi}(\bar{x}) - \tilde{\Phi}^*}{\varphi(\bar{x})}\;. \tag{2.7}$$

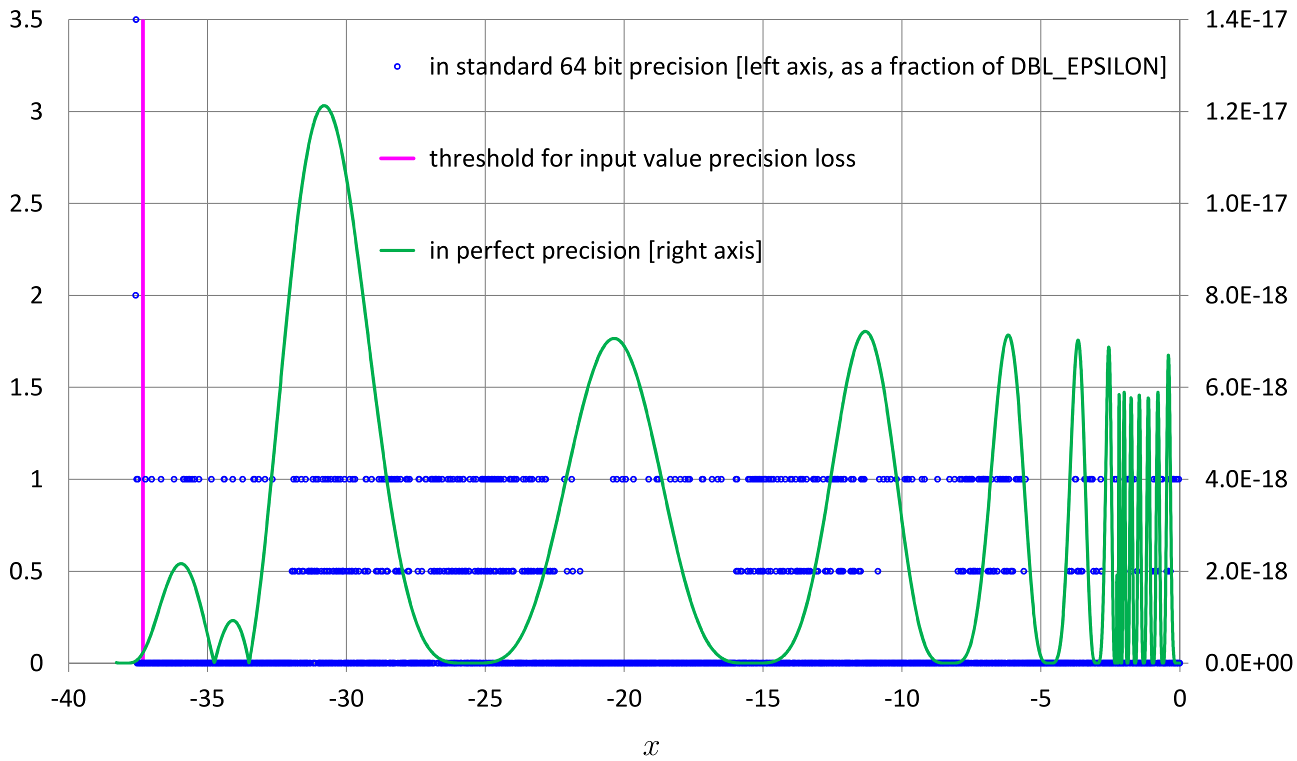

Figure 1: The absolute value of the relative accuracy in x of the analytical inversion formula for $\tilde{\Phi}(x)$.

The accuracy of this approximation is better than 1.22E-17 (in perfect precision). It is shown in figure 1 both for applied IEEE 754 double precision floating point calculations and in perfect arithmetic. We mention that the input value $\tilde{\Phi}^*$ becomes identically zero in IEEE 754 double precision for $x < -38.278$, and that it incurs loss of precision when $x < \tilde{\Phi}^{-1}(-\text{DBL\_MIN}) \approx -37.32$. In other words, the input value no longer has 15 digits of precision to match when the sought value for x in perfect precision is less than $\tilde{\Phi}^{-1}(-\text{DBL\_MIN})$, and that is why we see some of the numerical relative accuracy results (two blue dots) to the left of the graph show up at multiples of the general machine resolution given by DBL_EPSILON. This is not a failure of the presented algorithm but simply a loss of precision of the input value.

References

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