Options On Credit Default Index Swaps

Yunkang Liu and Peter Jäckel

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Abstract

The value of an option on a credit default index swap consists of two parts. The first one is the protection value due to potential default of the reference names before option expiry date. The second one depends on the value of the underlying credit default index swap with the index consists of the remaining survived names at the option expiry date. This article presents a one period tree-like valuation framework. As an example, the one-factor Gaussian Copula is used to model the state of the reference pool of the index at the option expiry date. Conditional on the state of the reference pool at the expiry date, the option is valued under the assumption that the conditional forward credit default index swap rate follows a displaced diffusion (shifted lognormal) process.

1 Introduction

A credit default swap on an index (CD index swap) is to some extent similar to a single name credit default swap (CDS). The index is a portfolio of defaultable reference names with equal weights. Like a single name CDS, a CD index swap consists of a protection leg and a premium leg. The cash flows on the protection leg are contingent on losses incurred from credit events of the reference names. The premium leg consists of scheduled coupon payments with a fixed coupon rate. The sizes of these cash flows depend on the recovery rate and the notional amount outstanding. For example, consider an index of 100 reference names with a total notional of 100 millions, if one name defaults then the notional amount for premium calculation will be reduced by one million, the protection buyer will deliver one million principal amount of the bonds of the reference name suffering the credit event to the protection seller in return for one millon in cash. A CD index swap usually carries a fixed coupon rate. When it is traded on the market at a different level, there is a requirement to pay or receive an up-front amount to enter a contract. For example, if an index has a fixed coupon of 100 basis points (bps) and it is traded at 110bps then the protection seller will receive an up-front payment equivalent to 10bps for the duration of the contract. This up-front payment is usually calculated using Bloomberg's CDS pricer.

Options on CD index swaps give investors the right to buy or sell risks at the strike spread. A payer swaption is an option that gives the holder the right to be the premium payer (protection buyer), while a receiver swaption holder has the right to be the premium receiver (protection seller). A payer swaption is often referred to as a put (right to sell risk), a receiver swaption as a call (right to buy risk). Also traded on the market is a straddle which entitles the holder to choose whether to buy or to sell risk at a specified strike spread. Options on CD index swaps are traded as knock-in. In the case of a payer swaption, should the option holder exercise it at the expiry date the holder will be compensated for losses incurred from any default of the reference names between the trade date and the expiry date. In the case of a receiver swaption, the option holder will have to pay for the losses should he exercise the option. By contrast, a swaption on a single name CDS is usually traded as knock-out, i.e. if the reference name defaults before the option expiry date the trade is terminated and there is no exchange of cash flows. A knock-in payer swaption is always more valuable than its knock-out version. For CD index swap, a knock-in receiver swaption is less valuable than its knock-out version.

This paper deals with the pricing of options on CD index swaps. Section 2 presents a one-period tree like approach, each tree node represent a state of the reference pool at the option expiry. By conditioning the option price on the state of the reference pool we have effectively followed the survival measure approach used by Schönbucher[Sch03] for pricing options on single name CD swaps. Section 3 presents an alternative pricing model that is currently used by some market practitioners.

2 A survival measure based option pricing model

Consider a forward-start CD index swap with a forward start date t_0, coupon payment dates $t_1, \ldots, t_n$, and a coupon rate of K. Without looking at details of all the reference names, we assume the reference pool is homogeneous. Therefore, we treat the index as if it were a single name for the purpose of making assumptions on the recovery rate and for bootstrapping the default probability from the market prices of traded index swaps. Furthermore, the interest rate and default intensity are assumed to be independent.

Denote

δ_i	the year fraction between t_{i-1} and t_i,
$D(t,u)$	the risk free discount factor, i.e., the price at t of a zero bond that pays 1 at u,
R	the recovery rate of a reference name,
$p(t,u)$	the default probability at time $u \geq t$ of a reference name conditional on survival at t

and for $t \leq t_0$ let

$$\bar{A}(t) = \sum_{i=1}^{n} \delta_i \left[D(t,t_i)(1-p(t,t_i)) + \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} (u - t_{i-1}) D(t,u)\, \mathrm{d}p(t,u) \right], \tag{1}$$

be the risky annuity[1], and

$$\bar{B}(t) = (1-R) \int_{t_0}^{t_n} D(t,u)\, \mathrm{d}p(t,u), \tag{2}$$

the expected value of the protection per unit notional.

Let us consider an option to enter the aforementioned CD index swap at the expiry date $T = t_0$. If there are defaults amongst the reference names before the option expiry date, the reference pool contains less reference names at the option expiry date T than at the valuation date. For this reason, the valuation of the option should be conditional on the state of the reference pool at the expiry date. Let N be the notional amount of each name, M the number of reference names remaining at the valuation date $t = 0$, and $\bar{L}$ the accumulated loss due to default between the trade date and the valuation date. Denote by Ω_m the set of exact m defaults in the reference pool between the valuation date and the option expiry date T. Conditional on Ω_m being realised at the expiry date T, the value at time $t \leq T$ of the aggregated loss due to default can be written as

$$L_m(t) = (\bar{L} + (1-R)mN) D(t,T), \tag{3}$$

[1] If the coupon is paid on the notional remaining at the coupon date, i.e., contingent accrued coupon is not paid, then the annuity should be $\bar{A}(t) = \sum_{i=1}^{n} \delta_i D(t,t_i)(1-p(t,t_i))$.

and the value of the underlying index swap to the protection buyer is

$$B_m(t) - KA_m(t), \tag{4}$$

where

$$A_m(t) = (M-m)N\tilde{A}(t), \qquad B_m(t) = (M-m)N\tilde{B}(t), \tag{5}$$

with

$$\tilde{A}(t) = \frac{1}{1-p(t,T)}\bar{A}(t), \qquad \tilde{B}(t) = \frac{1}{1-p(t,T)}\bar{B}(t) \tag{6}$$

being the annuity and protection value per unit notional conditional on the survival of the rest of the reference names at the expiry date T.

Let us consider the valuation of payer swaptions first. Conditional on the state of the reference pool being Ω_m at the expiry date T, the pay-off at T is

$$\max(L_m(T) + B_m(t) - KA_m(T), 0) = A_m(T)\max(S_m(T) - K, 0), \tag{7}$$

where

$$S_m(t) = \frac{L_m(t) + B_m(t)}{A_m(t)} \tag{8}$$

is the conditional default-adjusted forward CD index swap rate. For $0 \le m < M$, by using $A_m(t)$ as the numéraire, we see that the conditional present value of the pay-off is

$$V_m^{\text{payer}} = A_m(0)\,\mathbb{E}^Q[\max(S_m(T) - K, 0)], \tag{9}$$

where Q is the survival measure generated by the survival annuity $\tilde{A}(t)$. Trivially, we have

$$V_M^{\text{payer}} = L_M(0). \tag{10}$$

Therefore, the present value of the payer swaption is

$$V^{\text{payer}} = \sum_{m=0}^{M-1} P_m A_m(0)\,\mathbb{E}^Q[\max(S_m(T) - K, 0)] + P_M L_M(0), \tag{11}$$

where P_m denotes the probability of Ω_m. Similarly, we find the value of a receiver swaption as

$$V^{\text{receiver}} = \sum_{m=0}^{M-1} P_m A_m(0)\,\mathbb{E}^Q[\max(K - S_m(T), 0)]. \tag{12}$$

It can be seen from

$$\sum_{m=0}^{M} P_m = 1, \qquad \sum_{m=0}^{M} mP_m = Mp(t,T), \qquad \mathbb{E}^Q[S_m(T)] = S_m(0) \tag{13}$$

that the values of payer and receiver swaptions obey the so-called "call-put" parity

$$V^{\text{payer}} - V^{\text{receiver}} = MN\bar{A}(0)(\bar{S}(0) - K), \tag{14}$$

where

$$\bar{S}(t) = \frac{(\bar{L} + (1-R)MNp(t,T))D(t,T)}{MN\bar{A}(t)} + S(t) \tag{15}$$

is defined as the default-adjusted forward CD index swap rate with $S(t) = \bar{B}(t)/\bar{A}(t)$ being the (unadjusted) forward CD index swap rate. Because of this "call-put" parity we need only consider payer swaptions in the rest of this paper.

Empirical research such as [MW03] suggests that the dynamical behaviour of credit defaults swap rates tend to be complex and neither completely lognormal nor normal. In general it is close to normal distribution for high credit quality reference names and lognormal distribution for low quality reference names. Therefore, a reasonable assumption would be that the conditional default-adjusted forward CD index swap rate $S_m(t)$ follows the *displaced diffusion process*

$$\mathrm{d}S(t) = \sigma[\beta S(t) + (1-\beta)S(0)]\,\mathrm{d}W_t, \qquad t > 0 \tag{16}$$

where $\beta \geq 0$ is a displacement parameter and σ is a volatility parameter. Noting that there is no drift term in (16) because the swap rate we are concerned with is a martingale with respect to the probability measure Q. When $\beta = 1$, (16) is lognormal. When $\beta = 0$, it becomes a normal process. A brief discussion of the displaced diffusion process and its connection with CEV process in the context of implied volatility skew can be found in [Jäc02]. However, one needs to take extra care in the case of $\beta < 1$, since the swap rate can be negative with a small but positive probability. The beauty of using the displaced diffusion process is that it is no harder to get analytic option price formulae than it is in the case of lognormal process.

For simplicity, we assume β and σ are independent of Ω_m. Knowing that $\beta S(t) + (1-\beta)S(0)$ is lognormal when $\beta > 0$ and that $S(t)$ is normal when $\beta = 0$, we have

$$\mathbb{E}^Q[\max(S_m(T) - K, 0)] = \mathrm{Call}(S_m(0), K, T, \sigma, \beta), \tag{17}$$

where

$$\mathrm{Call}(S, K, T, \sigma, \beta) = \begin{cases} \left[\frac{\sigma\sqrt{T}S}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}d_3^2} - (K-S)\Phi(d_3)\right], & \beta = 0, \\ \frac{1}{\beta}\left[S\Phi(d_1) - K_\beta\Phi(d_2)\right], & \beta > 0, \end{cases} \tag{18}$$

with $K_\beta = \beta K + (1-\beta)S$ being the adjusted strike, and

$$d_1 = \frac{\log(S/K_\beta)}{\beta\sigma\sqrt{T}} + \frac{1}{2}\beta\sigma\sqrt{T}, \qquad d_2 = d_1 - \beta\sigma\sqrt{T}, \qquad d_3 = \frac{S-K}{\sigma\sqrt{T}S}. \tag{19}$$

In order to calculate the distribution of the number of defaults at the expiry T in a way that is consistent with the market practice for CDO tranche pricing, we choose the one-factor Gaussian Copula model (also known as the large pool approximation when the reference pool is homogeneous) to obtain

$$P_m = \int_{-\infty}^{\infty} \binom{M}{m} q(x)^m(1-q(x))^{M-m}\phi(x)\,\mathrm{d}x, \qquad m = 0, 1, \cdots, M, \tag{20}$$

where $\rho \geq 0$ is a correlation parameter[2], ϕ is the density function of the standard normal distribution, and

$$q(x) = \Phi\left(\frac{\Phi^{-1}(p(0,T)) - \sqrt{\rho}x}{\sqrt{1-\rho}}\right), \tag{21}$$

is the default probability at time T conditional on a given sample x of the standard normal distribution with Φ being its cumulative density function.

By combining formulae (11), (17) and (20) we are able to price payer swaptions on credit default indices. The resulting formula resembles the equity call option formula (see, e.g., [Jos03]) when the underlying process is assumed to be a jump diffusion of the form

$$\frac{\mathrm{d}S_t}{S_t} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + (J-1)\,\mathrm{d}N(t) \tag{22}$$

[2]For short-dated options, it is reasonable to assume that P_m is very small for large m. Hence, one may choose ρ to be the implied correlation of the traded equity tranches of the corresponding indices.

where J is the jump size, $N(t)$ is a Poisson process. Of course, in our case there are only a finite number of jumps with each jump corresponds to the default of a reference name, the sizes of these jumps are not identical, and the arrival of the jumps are not Poisson. It is worth noting that (11) is a generic pricing formula and one can choose different models for calculating P_m and $\mathbb{E}^Q[\max(S_m(T) - K, 0)]$.

Example 1. To illustrate the impact of the correlation parameter ρ on the option price, we consider payer swaptions that give the option holders the right to buy protection in three months time on a 4Year9Month index CD swap. Suppose the index consists of 30 reference names with a total notional of 30 million. It is traded at 100bps flat and is assumed to have 40% recovery rate. The interest swap rate is 4%. The model parameters are $\sigma = 50\%$ and $\beta = 1$. Figure 1 shows that correlation has little effect on the option value when it is not deeply out-of-money, i.e., when the strike K is not far greater than the default-adjusted forward CD index swap rate, which is $\bar{S}(0) = 106bps$ in this case.

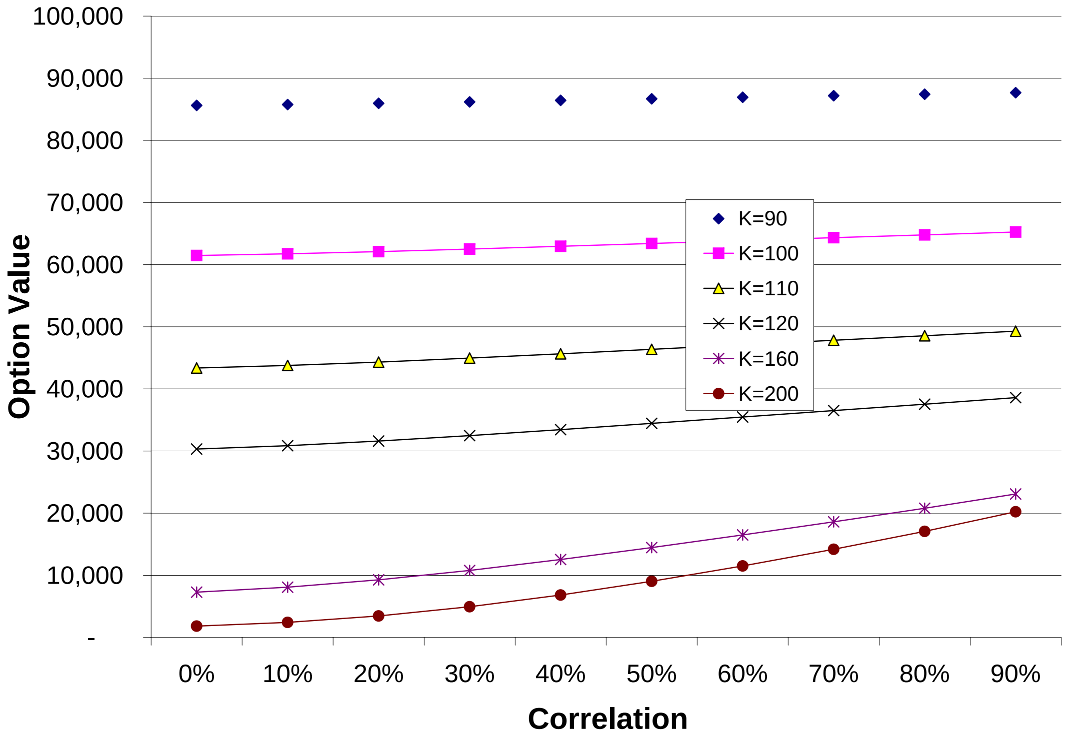

Figure 1: Sensitivity To Correlation

Example 2. When $\beta \neq 1$, the option pricing model present here can generate volatility skews, i.e., the implied Black volatility (the equivalent volatility for $\beta = 1$) can display certain levels of skewness. Consider the same put option as in Example 1 but priced with different values of β. Figure 1 shows that the two different shapes of skewness, one for $\beta < 1$ and the other for $\beta > 1$.

To model volatility smile, one may choose to replace the displaced diffusion process (16) with stochastic volatiltity models such Heston or SABR.

The ideas present in this section can be extended to cover the case of inhomogeneous reference pool[3]. However, we may have to make adjustment to the CDS curves of the underlying reference names. This is because, while we can infer the fair credit default swap rate (the intrinsic value) of

[3]The number of possible states of the reference pool is 2^M, which can be very large even for a relatively small M such as 30 (the number of names for the DJ iTraxx Europe HiVol and the Crossover indices). To be computationally practical, one may have to approximate the original reference pool by a homogeneous pool for cases where there are more than a couple of defaults.

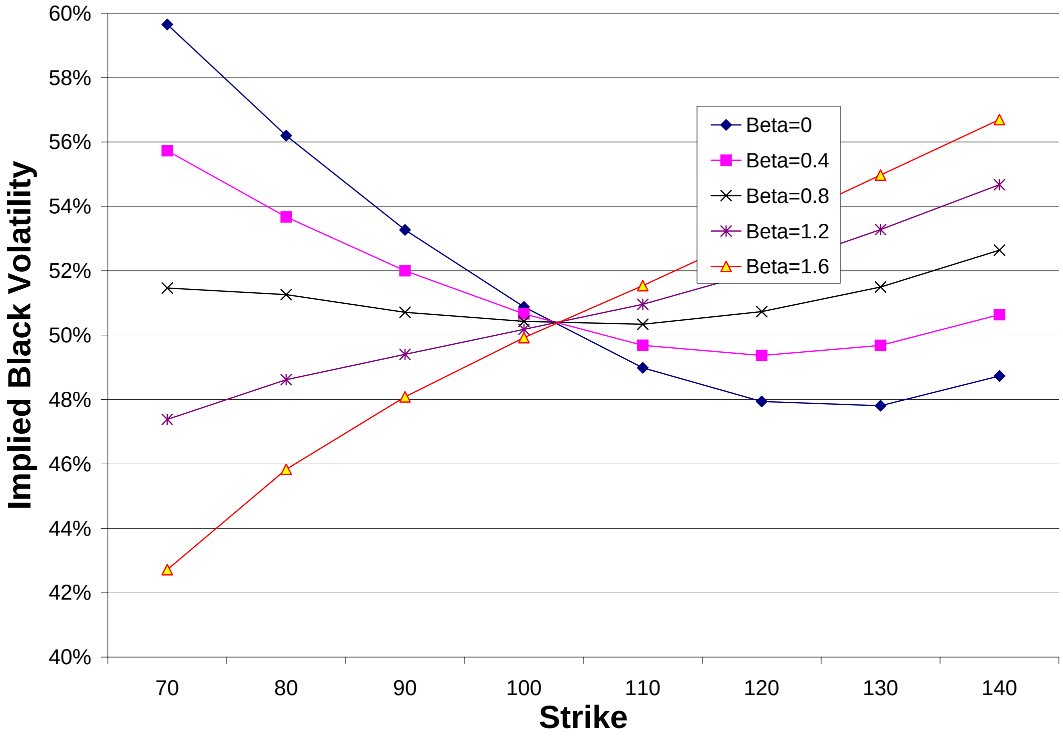

Figure 2: Volatility Skew

an index CD swap directly from the CDS curves of the underlying reference names, the market may trade the index swap at a different level due to issues such as liquidity and different definitions of credit event for indices and for single names.

3 An alternative approach

The expected pay off of a payer CD swaption at the expiry date T can be written as

$$MN\max(\bar{B}(T) - K\bar{A}(T), 0)) = MN\bar{A}(T)\max(\bar{S}(T) - K, 0)). \tag{23}$$

Assuming the default-adjusted forward rate $\bar{S}(t)$ follows the displaced diffusion process (16), by using $\bar{A}(t)$ as a "numéraire" we find that

$$V^{\text{payer}} = MN\bar{A}(0)\text{Call}(\bar{S}(0), K, T, \sigma, \beta), \tag{24}$$

The lognormal case, i.e., $\beta = 1$, seems to be quite popular with some market practioners. Because of its simplicity, (24) is a suitable candidate for quoting purpose. However it does have problems when used for option valuation.

First of all, there is a technical flaw in the derivation of (24) in that the risky annuity $\bar{A}(t)$ is not a valid numéraire since it contains the values of defaultable assets and it can become zero upon the default of all underlying assets. In the special case of an index with a single reference name ($M = 1$)the pricing formula (24) is inconsistent with the obvious choice

$$V^{\text{payer}} = N\bar{A}(0)\text{Call}(S(0), K, T, \sigma, \beta) + (1 - R)Np(0, T)D(0, T), \tag{25}$$

which is simply the option value in the knock-out case[4] plus the present value of the expected

[4]See, e.g., [HW02] and [Sch03], where Schöbucher's paper contains an excellent discussion on the survival measue approach for pricing single name CD swaption.

aggregated loss at the option expiry date. Even though the model parameters σ and β in (24) have slightly different meanings from those in (25), it does show that (24) is less than ideal.

Secondly, (24) may not be adequate in capturing the distributional nature of the number of default at the option expiry, especially when the opton is deeply out-of-money[5]. To be more precise, (24) may underestimates the values of out-of-money options as is shown by the following example.

Example 3. Consider a payer swaption that gives the option holder the right to buy protection at 200*bps* in three months time on a 4Year9Month CD index swap of two reference names with a notional of five million each. Assume that the interest rate is 4%, both reference names' CDS curves traded at 100*bps* flat and are assumed to have 40% recovery rate. This means that the probability of a single name default within 3 months is approximately $1 - \exp(-0.01/(1-0.4) * 0.25) \approx 0.42\%$, the forward CDS rate for the index is 100*bps*, the default-adjusted forward CDS rate is $\bar{S}(0) = 106bps$, the annuity is 4.15. Let us estimate the option value from first principles. If there is one default before the option expiry, the option holder would exercises the option to receives a protection payment of $(1 - 0.4) * 5,000,000 = 3,000,000$, at the same time makes a mark-to-market loss of $(200 - 100)/10000 * 4.15 * 5,000,000 = 207,500$ on the remaining CD index swap, assuming the CDS spread on the remaining survived name has not changed. The total value to the option holder would be $2,792,500$ in this case. If both names default then the value to the option holder would be even greater. Suppose defaults between the two reference names are independent. The probability of having at least one default is $1-(1-0.42\%)^2 \approx 0.838\%$. Therefore, the expected value to the option holder would be greater than $2,792,500 * 0.838\% \approx 23,400$. However, the swaption price given by (24) is approximately 260 if the lognormal volatility of the default-adjusted forward CD index swap rate is assumed to be $\sigma = 50\%$. To have the option price given by (24) exceed $23,400$, a lognormal volatility well above 100% is needed.

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[5] For easy of comparison, we call a payer CD index swaption out-of-money if the strike K is greater than the default-adjusted forward $\bar{S}(0)$.