Quanto Skew with stochastic volatility

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Abstract

In an extension to [Jäc09], we further investigate the performance of common quanto approximations in a context of stochastic volatility for both the asset and the FX process.

1 Introduction

A *quantity adjusting option*, or *quanto* for short, is a derivative contract that provides a payoff in a currency different from the natural quote currency of the underlying asset at a prearranged FX exchange rate. In a previous article [Jäc09], we recapitulated the precise definition of quanto options, reiterated the exact (model-free) valuation equations, listed the most common ad-hoc valuation adjustments for quantos, and demonstrated their performance in the context of a double displaced diffusion model. We also gave the exact valuation formula for that model, as well as a highly accurate analytical approximation for effective implied volatility in terms of adjusted displaced diffusion β and σ parameters. In numerical examples, we highlighted that all of the (tested) commonly used ad-hoc adjustments can give rise to significant valuation discrepancies for long maturities or high volatility environments.

The considered ad-hoc approximations for quanto option valuation were:-

DFAQ — read an implied volatility number from the domestic smile of S for strike K. Subsequently, price the quanto option with Black's formula replacing the domestic forward F by

$$F' = F\mathrm{e}^{\hat{c}}, \quad \text{using} \ \ \hat{c} = \hat{\sigma}_S \rho_{SQ} \hat{\sigma}_Q T \tag{1}$$

with $\hat{\sigma}_S$ being the at-the-money(forward) domestic implied volatility of S, and $\hat{\sigma}_Q$ analagously. This approach, in essence, attempts to transfer the domestic skew unaltered to quanto options, whence we refer to it as the *Domestic-Forward-ATM-Quanto* method, or *DFAQ* for short.

QFAQ — determine the effective volatility coefficient directly with the quanto-adjusted effective forward F' as given in (1). Subsequently, price the quanto option with Black's formula using the same adjusted forward F'. We refer to this approach as the *Quanto-Forward-ATM-Quanto*, or *QFAQ* for short. Note that if domestic volatility is marked as absolutely *sticky-strike*, then the looked up volatility is the same as the domestic volatility, and QFAQ gives the same price as DFAQ. If, however, domestic volatility is marked using a model or formula that requires the par forward strike as input, such as, for instance, the SABR formula (2.17a) in [HKL02], but also various other methods of marking domestic volatility, then QFAQ will give prices different from DFAQ. Where an actual model based on the forward is used, this approach can be seen as setting the initial value of the T-forward to F' in the money market measure.

QFAQ' — this is the same as QFAQ but with the exception that, where a model (formula) is used, the underlying asset price is assumed to have initial value as seen today, and its relative instantaneous drift is increased by $\hat{c} = \hat{\sigma}_S \rho_{SQ} \hat{\sigma}_Q$ to ensure that the model's expected risk-neutral T-forward is equal to F' in the money market measure. Note that methods QFAQ and QFAQ' are mathematically equivalent for models whose local volatility component is purely linear as for geometric Brownian motion.

A question that remained open was whether the poor performance of commonly used ad-hoc adjustments is due to the intrinsic nature of the employed model belonging to the family of parametric local volatility models. In this article, to complement the previous research, we revisit the quanto adjustment to vanilla options in the context of a model that allows for both local and stochastic volatility. For reasons of fundamental economical features, such as temporal decorrelation of instantaneous volatility, and unattainability of zero for the asset price, as well as for reasons of numerical stability, we use the hyperbolic-local-hyperbolic-stochastic model, *HypHyp* for short, first suggested in [JK07, JK08] as the fundamental building block of quanto valuation.

2 Quanto valuation with the double HypHyp model

In aid of numerical valuation, we denote the quanto model directly in logarithmic coordinates according

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to

$$
\begin{aligned}
\mathrm{d}\xi_S &= \left[r_S - d_S - \tfrac{1}{2}(\sigma_S f_S g_S)^2\right]\mathrm{d}t + \sigma_S f_S g_S \cdot \mathrm{d}W_S \\
\mathrm{d}y_S &= -\kappa_S y_S \mathrm{d}t + \alpha_S\sqrt{2\kappa_S}\cdot \mathrm{d}Z_S \\
\mathrm{d}\xi_Q &= \left[r_S - r_Q - \tfrac{1}{2}(\sigma_Q f_Q g_Q)^2\right]\mathrm{d}t + \sigma_Q f_Q g_Q \cdot \mathrm{d}W_Q \\
\mathrm{d}y_Q &= -\kappa_Q y_Q \mathrm{d}t + \alpha_Q\sqrt{2\kappa_Q}\cdot \mathrm{d}Z_Q
\end{aligned}
\tag{2}
$$

and

$$
\begin{aligned}
f_S &= f(\mathrm{e}^{\xi_S}) & f_Q &= f(\mathrm{e}^{\xi_Q}) \\
g_S &= g(y_S) & g_Q &= g(y_Q) \\
S &= S_0\cdot \mathrm{e}^{\xi_S} & Q &= Q_0\cdot \mathrm{e}^{\xi_Q}
\end{aligned}
\tag{3}
$$

in the money market measure nominated in the quotation currency of the asset S, allowing for the short interest rate r_S and convenience yield rate d_S. This model is based on the separation of the local volatility component $f(\cdot)$ and the stochastic volatility component $g(\cdot)$ with the functions

$$
f(x) = \frac{1}{\beta}\Big[(1-\beta+\beta^2)\cdot x + (\beta-1)\cdot\big(\sqrt{x^2+\beta^2(1-x)^2}-\beta\big)\Big] \tag{4}
$$

$$
g(y) = y + \sqrt{y^2+1}\,. \tag{5}
$$

The choice of these functions is discussed in detail in [Jäc06, JK07, JK08]. Here, suffice it to say that $f(\cdot)$ can be seen as an approximation to the CEV model with the difference that zero is unattainable, and $g(\cdot)$ as an approximation to the Wiggins/Scott/Scott-Chesney/Hull-White/SABR model for stochastic volatility with geometric Brownian motion [Wig87, Sco87, CS89, HW88, HKL02] with the difference of e^y being replaced by $y+\sqrt{y^2+1}$ (which matches e^y to second order in y). Specifically with respect to $f(\cdot)$, we point out that if the asset part of this model is used with the aforementioned ad-hoc quanto adjustments, then methods QFAQ and QFAQ' are mathematically equivalent when $\beta_S = 1$. To be precise, method QFAQ would mean to set $r_S \to 0$, $d_S \to 0$, and $S_0 \to F'$ before numerically pricing the option struck at K (which still has to be discounted), whereas QFAQ' means to set $d_S \to d_S - \hat\sigma_S\rho_{SQ}\hat\sigma_Q$ to attain the same risk-neutral forward, and then numerically price the option struck at K. Naturally, when the quanto considerations are with respect to interest rates, all previously mentioned drift adjustments need to be amended with respect to the specific interest rate model used.

Irrespective of the specific form of the convection-diffusion terms in the given stochastic differential equations, generic valuation of vanilla quanto options is governed by a partial differential equation of the form

$$
\partial_t v + \sum_i \mu_i \partial_{x_i} v + \tfrac{1}{2}\sum_{i,j}\varsigma_i\rho_{ij}\varsigma_j\partial_{x_i x_j} v = rv \tag{6}
$$

with terminal conditions

$$
v(T) = (\theta(S_T - K))_+ \cdot Q_T \tag{7}
$$

with $\theta = 1$ for calls and $\theta = -1$ for puts. The valuation with (6) and (7) gives the value of the quanto option expressed in currency units of the trading currency of asset S. If desired, this price can of course be translated into the quanto currency (recalling that Q denotes the value of one quanto currency unit expressed in the asset currency) by division by Q_t. Since this comprises a simple linear scaling, and since, in the following, we use numerical examples with $Q_t = 1$, we omit the mentioning of the term $1/Q_t$ throughout.

The generic coordinates $\{x_i\}$ in (6) for the double HypHyp model are of course to be read as $\{\xi_S, y_S, \xi_Q, y_Q\}$, and the associated convection coefficients μ_i and effective diffusion coefficients ς_i are obviously given by the terms proportional to $\mathrm{d}t$, and $\mathrm{d}W_{(\cdot)}$ or $\mathrm{d}Z_{(\cdot)}$, respectively, as seen in equation (2). The source term coefficient on the right hand side of (6) is naturally $r \equiv r_S$ since we are pricing in the money market measure in the trading currency of S.

Whilst analytical approximations for vanilla options priced with the standard HypHyp model are available [JK07, JK08], for the purposes of this article, we rely solely on numerical calculations. Specifically, we used an explicit finite differencing solver in four dimensions for all calculations.

It remains to be said that the set of parameters σ_S, α_S, β_S, κ_S, ρ_S, r_S, d_S, and σ_Q, α_Q, β_Q, κ_Q, ρ_Q, and r_Q are all assumed to be determined exclusively from the domestic forward contract and vanilla option market, allowing for a very flexible parametrisation of the observable implied volatility smiles.

2.1 Correlation

The double HypHyp model allows for correlation between the underlying asset's driving Wiener process $\mathrm{d}W_S$ and its volatility driver $\mathrm{d}Z_S$. The correlation between these two diffusions is denoted as

$$
\rho_S = \mathsf{E}\left[\mathrm{d}W_S \mathrm{d}Z_S\right]/\mathrm{d}t\,, \tag{8}
$$

and ρ_Q is defined in complete analogy. Also, the model permits correlation between the asset process driver and the FX rate process driver

$$
\rho_{SQ} = \mathsf{E}\left[\mathrm{d}W_S \mathrm{d}W_Q\right]/\mathrm{d}t \tag{9}
$$

and this quantity is assumed to be given as a further input. With this, we can depict the auto-correlation matrix of the state vector $\boldsymbol{x} = (\xi_S, y_S, \xi_Q, y_Q)^\top$ in the form

$$
\mathsf{E}\left[\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}^\top\right]/\mathrm{d}t = \begin{pmatrix} 1 & \rho_S & \rho_{SQ} & \rho_{14} \\ \rho_S & 1 & \rho_{23} & \rho_{24} \\ \rho_{SQ} & \rho_{32} & 1 & \rho_Q \\ \rho_{41} & \rho_{42} & \rho_Q & 1 \end{pmatrix}. \tag{10}
$$

While it may be reasonable to expect the asset/FX correlation to be numerically attainable from time series, having access to reliable estimates for the inter-volatility correlations, and cross-asset-volatility correlations may be a far stretch. For this reason, we fill in the remaining fields by the aid of the parametric form

$$\begin{aligned} \rho_{14} &= \rho_{41} = \rho_{SQ} \cdot \rho_Q \\ \rho_{23} &= \rho_{32} = \rho_{SQ} \cdot \rho_S \\ \rho_{24} &= \rho_{42} = \rho_S \cdot \rho_{SQ} \cdot \rho_Q + \beta_\rho \cdot \sqrt{1-\rho_S^2} \cdot \sqrt{1-\rho_Q^2} \end{aligned} \tag{11}$$

with $\beta_\rho \in [-1, 1]$ as suggested in [JK09]. Note that whilst the choice (11) uniquely determines the cross-asset-volatility correlations, it leaves open a comparatively wide range of volatility-volatility correlations. It is worth noting that this underspecification is an intrinsic issue with stochastic volatility models as soon as more than one financial observable is involved in the model's configuration. We will see later what impact the flexibility in volatility correlations has on the uncertainty of quanto option prices.

3 Numerical examples

In [Jäc09], numerical results were presented showing the DFAQ and QFAQ approximations in comparison to exact quanto option values in the framework of a double displaced diffusion model. The presented figures were all given both in terms of time-values (option price minus intrinsic value) and in terms of implied volatilities. Implied volatilities were computed for each curve with the most relevant effective par forward for that curve, respectively.

Here, we will instead show all quanto curves as implied volatilities backed out from the quanto option prices using one and the same "exact" quanto forward $\tilde{F}$ which is taken from the fully fledged four-dimensional numerical quanto calculation. We will show the domestic smile of the asset S, the FX smile of Q, and four different quanto smiles. The curve denoted as "exact" represents the full numerical calculation in four dimensions. We should point out that whilst the finite differencing solution may not be close to the exact solution of the double HypHyp equations down to the last digit of machine accuracy, the explicit finite differencing calculation is at all times a self-consistent model in its own right. To see this, the reader is reminded that an explicit finite differencing calculation is nothing other than a conventional tree with explicitly calculated transition probabilities, and certain lateral boundary conditions[1]. Thus, even if a finite difference to an idealised exact solution of the original equations could be detected (though we believe that our results are numerically highly accurate and well converged), the used numerical model is still a valid and arbitrage-free model regardless, just like the Black-Derman-Toy [BDT90] and the Black-Karasinski [BK91] model are, in their discrete tree setting, perfectly legitimate models.

Before we begin reporting the numerical results, we highlight that, due to the fact that DFAQ, QFAQ, and QFAQ' prices are all computed with effective forwards that are away from the "exact" quanto forward $\tilde{F}$ (which we establish numerically), when DFAQ, QFAQ, and QFAQ' prices are converted to implied volatilities using $\tilde{F}$, call and put prices, for DFAQ, QFAQ, and QFAQ', will not have the same implied volatilities for the same strike. This may at first feel somewhat counterintuitive, and awkward for like-for-like comparisons, but it is a consequence that we have to deal with, one way or another. For the purpose of identification of call versus put lines, we mention that in the following, in all diagrams, for disconnected lines (DFAQ, QFAQ, and QFAQ'), the call option branch is always the one that continues to the upper end of the abscicssa.

3.1 Medium dated quanto options

The first example we present is for maturity $T = 2$. Its full details are given in parameter set 1. The implied volatilities for this parameter set are shown in figure 1. We can see that, while there are some differences, overall, the domestic, DFAQ, QFAQ, QFAQ', and exact quanto curves are all fairly close together, and the discrepancy between the approximate and the exact quanto forward is (arguably) negligible.

	$T=2$					
X	X_0	σ_X	β_X	α_X	κ_X	ρ_X
S	1	15%	$\frac{3}{4}$	$\frac{1}{2}$	1	$-\frac{3}{4}$
Q	1	15%	1	$\frac{1}{2}$	1	0
	$r_S = d_S = r_Q = 0$, $\rho_{SQ} = -\frac{1}{2}$, and $\beta_\rho = 1$					

Parameter set 1

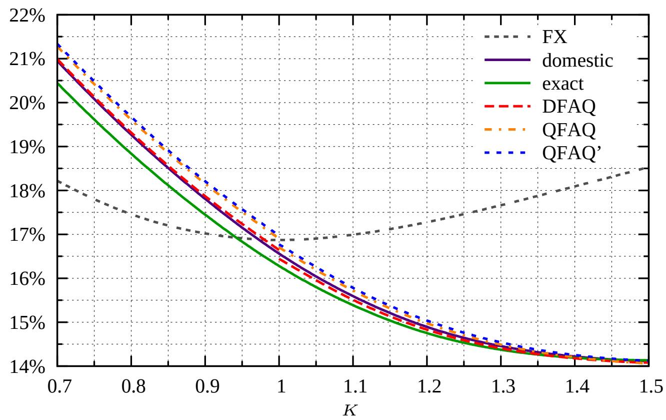

Figure 1: Domestic, FX, and quanto volatilities for parameter set 1. The approximate quanto forward is $F' = 1.0283$ and the (numerically computed) "exact" quanto forward is $\tilde{F} = 1.0295$.

Next, we have an example with a more pronounced smile and skew for the same maturity. The base pa-

[1] We used no-convection/no-diffusion boundary conditions, which amount to "no flux", or "reflective" boundary conditions. These conditions, strictly speaking, violate the local no-arbitrage conditions on the boundary but, since we had the boundaries significantly far out, we believe that this had no material impact on any of the results.

rameters are given in parameter set 2, and the numerical results are shown in figures 2 to 3 for different

	$T=2$					
X	X_0	σ_X	β_X	α_X	κ_X	ρ_X
S	1	7%	$\frac{3}{4}$	5	$\frac{1}{4}$	$-\frac{3}{4}$
Q	1	7%	1	4	$\frac{1}{4}$	0
	$r_S = d_S = r_Q = 0$ and $\beta_\rho = 1$					

Parameter set 2

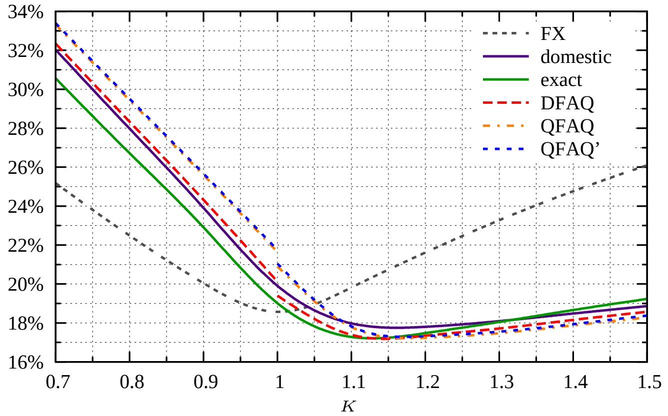

Figure 2: Implied volatilities for parameter set 2 with $\rho_{SQ} = \frac{1}{2}$. $F' = 1.0377$ and $\tilde{F} = 1.04505$.

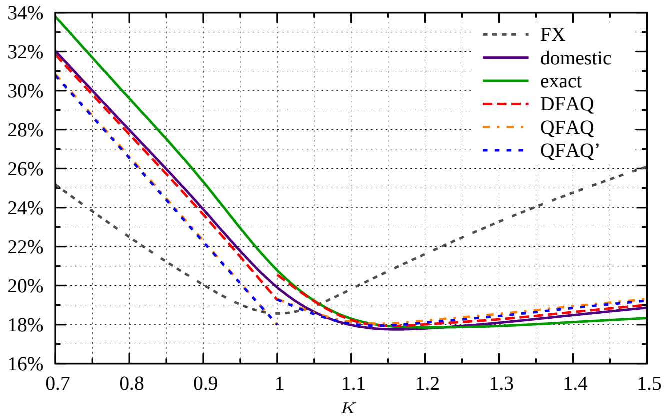

Figure 3: Implied volatilities for parameter set 2 with $\rho_{SQ} = -\frac{1}{2}$. $F' = 0.9637$ and $\tilde{F} = 0.9599$.

values of ρ_{SQ}. We can see that near the money and for put options, depending on the asset-FX correlation, the exact quanto smile is either below ($\rho_{SQ} = \frac{1}{2}$) or above ($\rho_{SQ} = -\frac{1}{2}$) the approximately adjusted quanto smiles. In addition, it is also visible that the difference between the QFAQ and QFAQ' methods is barely noticeable. This is in agreement with the fact that for this parameter set only the asset process S has a non-linear local volatility component with $\beta_S = \frac{3}{4}$, which is very mild considering that for $\beta_S = 1$ we expect no differences between QFAQ and QFAQ' other than numerical inaccuracies. Crucially, though, we notice that the DFAQ method appears to be closest to the exact solution, and that the magnitude of difference appears to be (again arguably) still acceptable, though larger than those with parameter set 1.

3.2 Long dated quanto options

We now look at an example with ten years to option expiry. All common parameters are shown in parameter set 3, and the numerical results are in figures 4 to 7. We notice that the overall behaviour

	$T=10$					
X	X_0	σ_X	β_X	α_X	κ_X	ρ_X
S	1	14%	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{1}{20}$	$-\frac{1}{2}$
Q	1	13%	1	1	$\frac{1}{10}$	$-\frac{1}{4}$
	$r_S = d_S = r_Q = 0$					

Parameter set 3

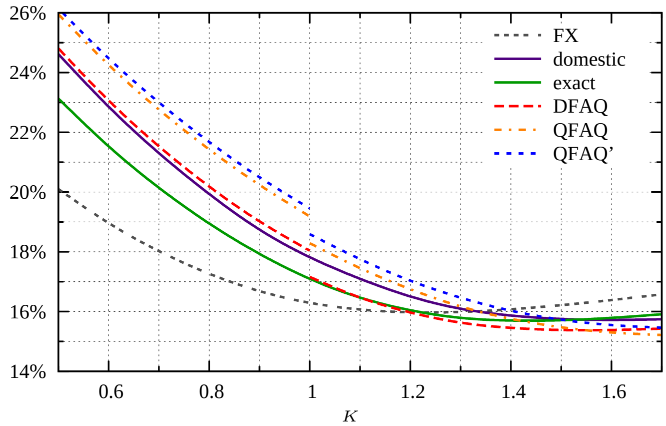

Figure 4: Implied volatilities for parameter set 3 with $\rho_{SQ} = \frac{1}{2}$, and $\beta_\rho = 1$. $F' = 1.1562$ and $\tilde{F} = 1.1696$.

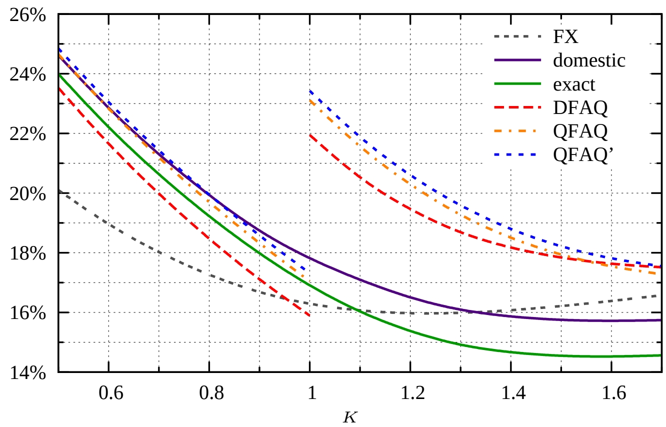

Figure 5: Implied volatilities for parameter set 3 with $\rho_{SQ} = \frac{1}{2}$, and $\beta_\rho = -1$. $F' = 1.1562$ and $\tilde{F} = 1.0832$.

shown in figures 4 and 6 is very similar to figures 2 and 3 with perhaps about 1% difference in implied volatility for put options between DFAQ (which performs best) and the exact solution. We also notice a slight widening between QFAQ and QFAQ' which we attribute to this example simply being longer dated than parameter set 2. However, if we look at figures 5 and 7, we observe a significant widening of the curves. This is also reflected in a greater difference between F' and $\tilde{F}$ (of about 6%) in comparison to figures 4 and 6 (where it was about 1%). We see here for the first time the effect caused by the choice of $\beta_\rho = -1$.

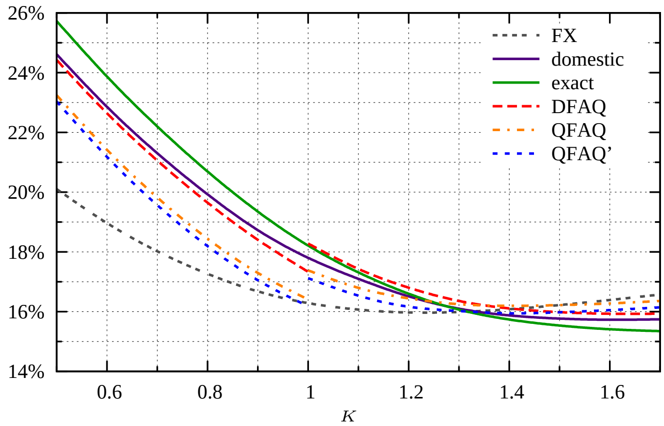

Figure 6: Implied volatilities for parameter set 3 with $\rho_{SQ} = -\frac{1}{2}$, and $\beta_\rho = 1$. $F' = 0.8651$ and $\tilde{F} = 0.8568$.

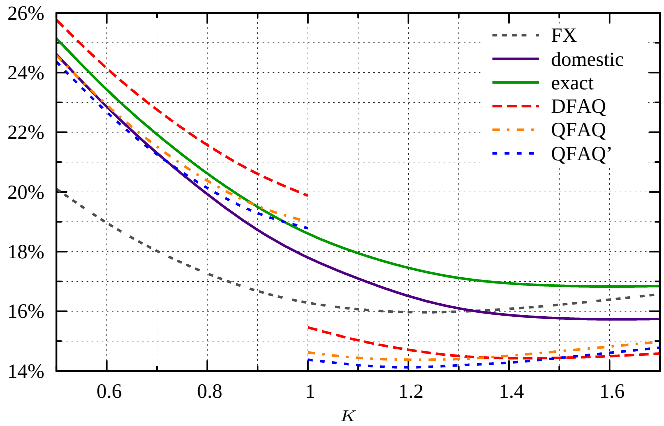

Figure 7: Implied volatilities for parameter set 3 with $\rho_{SQ} = -\frac{1}{2}$, and $\beta_\rho = -1$. $F' = 0.8651$ and $\tilde{F} = 0.9177$.

To understand this better, we give the full correlation matrices associated with figures 4 and 5:

$$\text{[figure 4]} \quad \begin{pmatrix} 1 & -0.5 & 0.5 & -0.125 \\ -0.5 & 1 & -0.25 & 0.901 \\ 0.5 & -0.25 & 1 & -0.25 \\ -0.125 & 0.901 & -0.25 & 1 \end{pmatrix} \tag{12}$$

$$\text{[figure 5]} \quad \begin{pmatrix} 1 & -0.5 & 0.5 & -0.125 \\ -0.5 & 1 & -0.25 & -0.776 \\ 0.5 & -0.25 & 1 & -0.25 \\ -0.125 & -0.776 & -0.25 & 1 \end{pmatrix} \tag{13}$$

The highlighted numbers are, of course, the volatility-volatility correlation which switches from 90.1% to −77.6%. Even though the instantaneous driver process volatility of the asset S and the FX rate Q is the same at 50% for these two examples, the change in inter-volatility correlation has the net effect of *terminal decorrelation*. As a consequence, the exact quanto forward $\tilde{F}$ is much closer to the domestic forward in the example in figure 5 than in that of figure 4. Since the ad-hoc quanto adjustment formulae simply used the instantaneous asset and FX process correlations, the net effect of terminal decorrelation due to anti-correlation of volatilities is underestimated in the example in figure 5. This reminds us that the fundamental quantity of importance for the assessment of quanto adjustments is the terminal covariance between asset and FX rate, not just the instantaneous process correlations.

For the sake of completeness, we also give the full correlation matrices associated with figures 6 and 7:

$$\text{[figure 6]} \quad \begin{pmatrix} 1 & -0.5 & -0.5 & 0.125 \\ -0.5 & 1 & 0.25 & 0.776 \\ -0.5 & 0.25 & 1 & -0.25 \\ 0.125 & 0.776 & -0.25 & 1 \end{pmatrix} \tag{14}$$

$$\text{[figure 7]} \quad \begin{pmatrix} 1 & -0.5 & -0.5 & 0.125 \\ -0.5 & 1 & 0.25 & -0.901 \\ -0.5 & 0.25 & 1 & -0.25 \\ 0.125 & -0.901 & -0.25 & 1 \end{pmatrix} \tag{15}$$

Here, too, we see that whereas we have strongly positive inter-volatility correlation in the example of figure 6, we have strongly negative inter-volatility correlation in the example of figure 7, and this in turn gives rise to the exact quanto forward being much closer to the domestic forward in figure 7 than in figure 6, with the same effect of the ad-hoc adjustments overestimating (absolute) terminal correlation of asset and FX rate.

3.3 Very long dated quanto options

The next example has twenty years to option expiry, and its common parameters are shown in parameter set 4. The numerical results are in figures 8 to 11. We see a very similar pattern to that observed in section 3.2, only that all numerical differences between approximations and the exact solution are now significantly larger, owing to the longer maturity. For twenty years to maturity, quanto options are certainly no longer vanilla even if one believes one has sufficient information for the modelling of the underlying asset and FX rate for that maturity.

	$T = 20$					
X	X_0	σ_X	β_X	α_X	κ_X	ρ_X
S	1	12%	1	2	$\frac{1}{20}$	$-\frac{3}{4}$
Q	1	5%	1	3	$\frac{1}{10}$	$\frac{1}{4}$
	$r_S = d_S = r_Q = 0$					

Parameter set 4

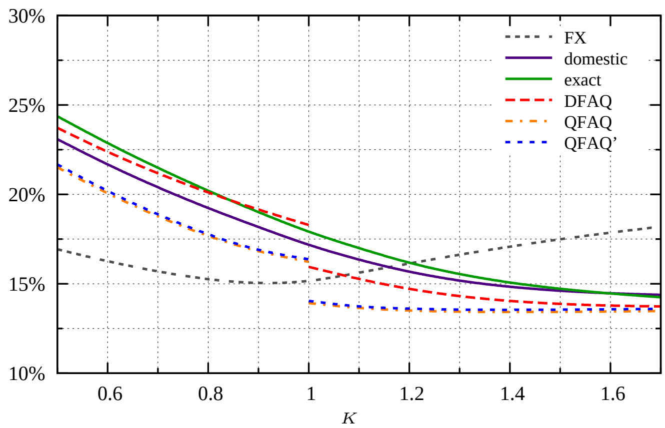

Figure 8: Implied volatilities for parameter set 4 with $\rho_{SQ} = -\frac{1}{2}$, and $\beta_\rho = 1$. $F' = 0.7706$ and $\tilde{F} = 0.8039$.

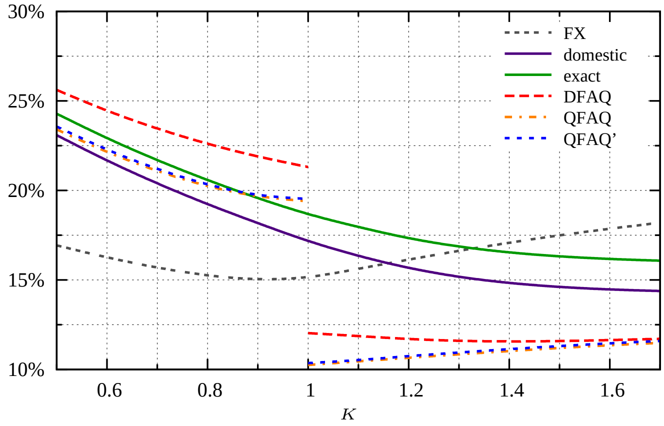

Figure 9: Implied volatilities for parameter set 4 with $\rho_{SQ} = -\frac{1}{2}$, and $\beta_\rho = -1$. $F' = 0.7706$ and $\tilde{F} = 0.9170$.

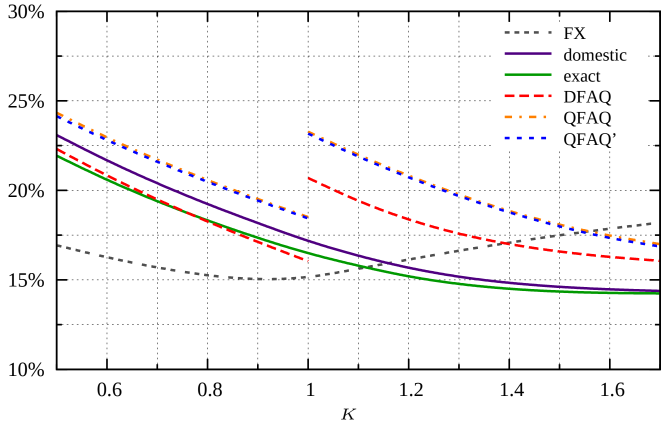

Figure 10: Implied volatilities for parameter set 4 with $\rho_{SQ} = \frac{1}{2}$, and $\beta_\rho = 1$. $F' = 1.2977$ and $\tilde{F} = 1.2159$.

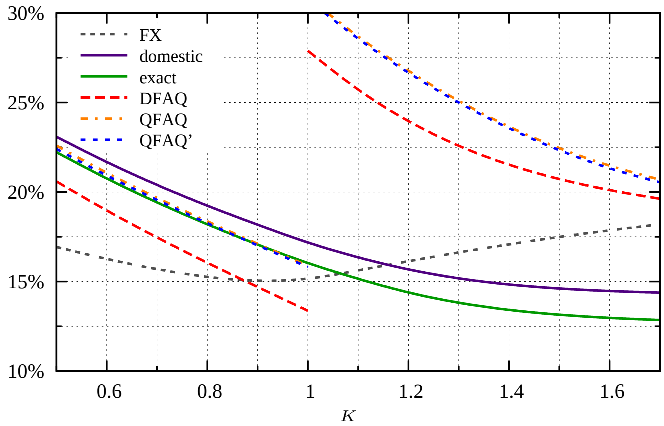

Figure 11: Implied volatilities for parameter set 4 with $\rho_{SQ} = \frac{1}{2}$, and $\beta_\rho = -1$. $F' = 1.2977$ and $\tilde{F} = 1.0592$.

3.4 A scary scenario

The final example we show has ten years to option expiry, though the parameters, as shown in parameter set 5, have been pushed up somewhat as one might observe in some asset classes such as commodities, for instance. The numerical results are in figures 12 to 17.

The six shown figures split into two groups. The first three figures 12 to 14 are for $\rho_{SQ} = -\frac{1}{2}$ with the volatility-volatility correlation scaling parameter β_ρ going through -1, 0, and 1. As in all previous examples, we see that for $\beta_\rho = -1$, the exact quanto for-

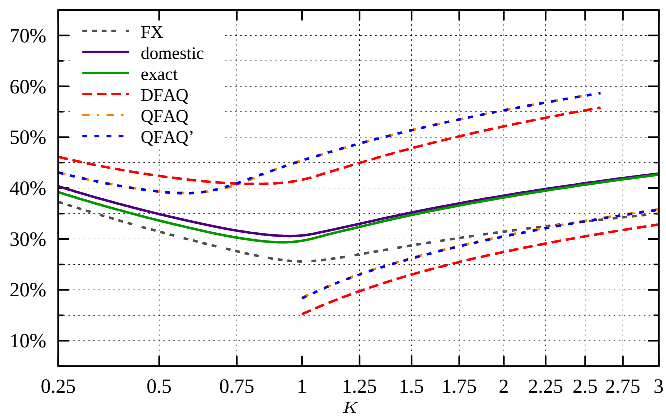

Figure 12: Implied volatilities for parameter set 5 with $\rho_{SQ} = -\frac{1}{2}$, and $\beta_\rho = -1$. $F' = 0.675$ and $\tilde{F} = 0.969$.

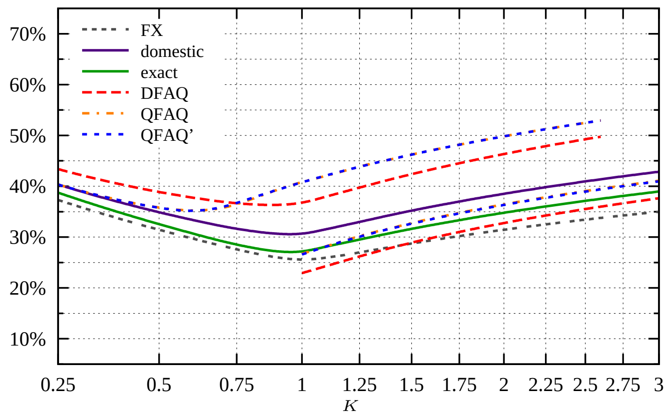

Figure 13: Implied volatilities for parameter set 5 with $\rho_{SQ} = -\frac{1}{2}$, and $\beta_\rho = 0$. $F' = 0.675$ and $\tilde{F} = 0.812$.

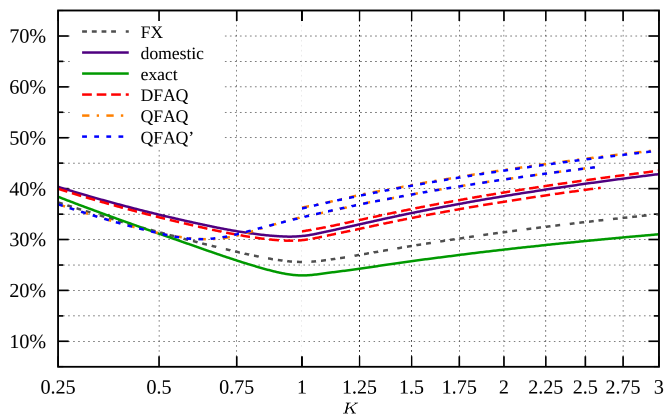

Figure 14: Implied volatilities for parameter set 5 with $\rho_{SQ} = -\frac{1}{2}$, and $\beta_\rho = 1$. $F' = 0.675$ and $\tilde{F} = 0.660$.

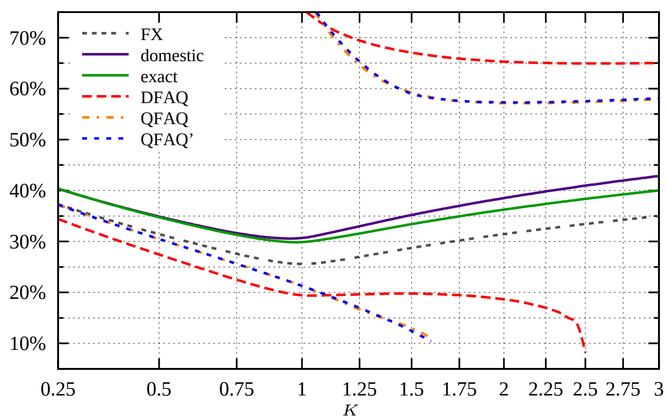

Figure 15: Implied volatilities for parameter set 5 with $\rho_{SQ} = \frac{1}{2}$, and $\beta_\rho = -1$. $F' = 1.481$ and $\tilde{F} = 0.963$.

	$T=10$					
X	X_0	σ_X	β_X	α_X	κ_X	ρ_X
S	1	8%	1	5	$\frac{1}{10}$	$\frac{1}{4}$
Q	1	7%	1	5	$\frac{1}{10}$	0
	$r_S = d_S = r_Q = 0$					

Parameter set 5

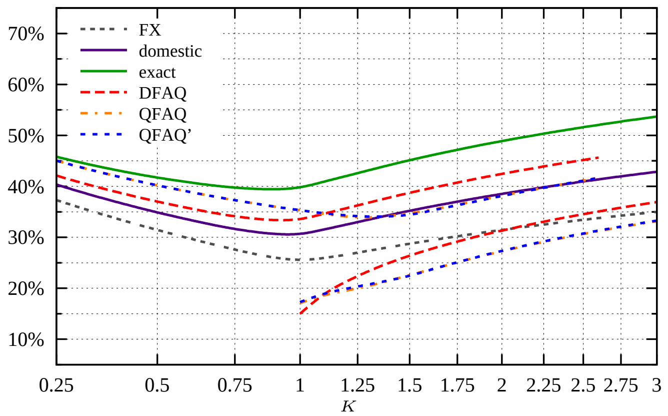

Figure 16: Implied volatilities for parameter set 5 with $\rho_{SQ} = \frac{1}{2}$, and $\beta_\rho = 0$. $F' = 1.481$ and $\tilde{F} = 1.695$.

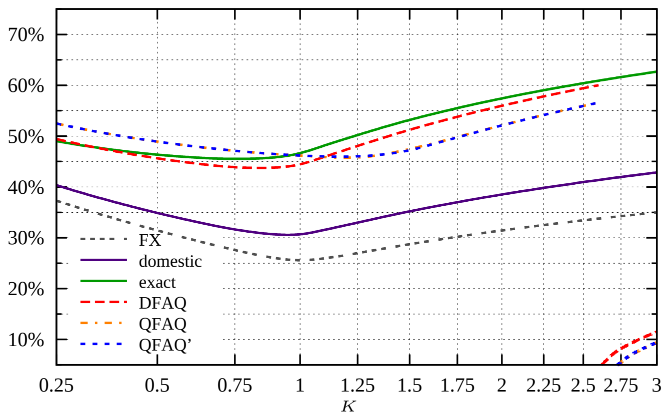

Figure 17: Implied volatilities for parameter set 5 with $\rho_{SQ} = \frac{1}{2}$, and $\beta_\rho = 1$. $F' = 1.481$ and $\tilde{F} = 3.092$.

ward $\tilde{F}$ is much closer to the domestic asset forward than the approximate quanto forward F'. However, here, for the first time, we see that as β_ρ is increased from -1, to 1, the exact quanto forward crosses and goes beyond the level the approximate quanto forward, albeit only ever so slightly as $\beta_\rho \to 1$. This phenomenon becomes exacerbated in figures 15 to 17 which are for $\rho_{SQ} = \frac{1}{2}$. We summarize in figure 18 the associated dependence of $\tilde{F}$ on β_ρ in comparison to F'. Note that the curve for $\tilde{F}(\rho_{SQ} = \frac{1}{2})$ extends *below* 1 for $\beta_\rho \to -1$, implying net terminal correlation between S_T and Q_T being negative for these parameter values. This is no artefact: it can indeed be shown that the net spot-FX correlation can be negative even when $\rho_{SQ} > 0$ provided that β_ρ is sufficiently negative *and* at least one of spot or FX has non-zero correlation with their own associated volatility, though the proof for this is beyond the scope of this article.

As a final point of note in this section, we draw the reader's attention to the scale of the volatility

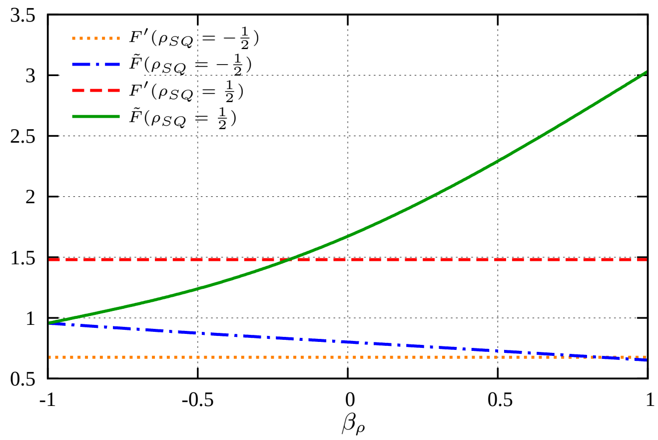

Figure 18: $\tilde{F}$ as a function of β_ρ in comparison to F' for parameter set 5. The apparent convergence of $\tilde{F}(\rho_{SQ} = -\frac{1}{2})$ and $\tilde{F}(\rho_{SQ} = \frac{1}{2})$ for $\beta_\rho \to -1$ is pure coincidence.

diagrams 12 to 17 which range from 5% to 75%. This, and the fact that in figure 17 the DFAQ and QFAQ/QFAQ' curves could not even be computed for call options with strikes less than (approximately) 2.5 serves as a stark reminder that long dated quanto options, especially in an environment of noticeable volatiliy, can turn out to be rather toxically model-dependent indeed.

4 Quanto forward matching

We saw in the previous section that, in the context of stochastic volatility models, the specification of instantaneous asset and FX process correlation alone leaves the level of the quanto forward widely underdetermined. Of crucial importance for the net correlation between the future asset and FX spots is also the magnitude of volatility-volatility correlation. This effect is significant when volatilities are sizeable, asset-FX correlation is positive, or maturities are reasonably long dated. This highlights that rather than providing an estimate for inter-process correlation, it is more sensible to given an idea of correlation of the terminal spot (asset and FX) realisations. The immediately obvious financial quantity to choose as a measure for terminal spot and FX correlation is of course the quanto forward itself. However, this still leaves open the question: given the quanto forward, are quanto option prices well determined? In other words, given that in a stochastic volatility model the quanto forward is not only a function of one instantaneous correlation parameter, but of several, we would like to have a feeling for the residual uncertainty.

More formally, consider one specific choice of process parameters for the model specified in equations (2)–(5). Using the correlation parametrisation defined in (11), and keeping all other parameters apart from ρ_{SQ} and β_ρ fixed, the quanto forward for a given expiry T is a certain (unknown) function, say ϕ, of

ρ_{SQ} and β_ρ:

$$\tilde{F}(t,T) = \mathsf{E}_t\left[S_T\frac{Q_T}{Q_t}\right] = \phi(\rho_{SQ},\beta_\rho)\ . \qquad (16)$$

If a target value F' for the quanto forward is specified, the constraint of matching it implicitly specifies ρ_{SQ} as a function of β_ρ by virtue of the implicit function theorem:

$$\rho_{SQ} = \rho_{SQ}(\beta_\rho; F')\ . \qquad (17)$$

For specific stochastic volatility models, analytical approximations for this function may be available. In this study, however, in keeping with its overall numerical spirit, we define it implicitly as a root-finding solution of (16).

To pick two examples, we choose $F' = 0.85$ and $F' = 1.25$ using parameter set 5, as we had in figures 12 to 17. We show the curves $\rho_{SQ}(\beta_\rho; F')|_{F'=1.25}$ and $\rho_{SQ}(\beta_\rho; F')|_{F'=0.85}$ in figure 19. We notice that

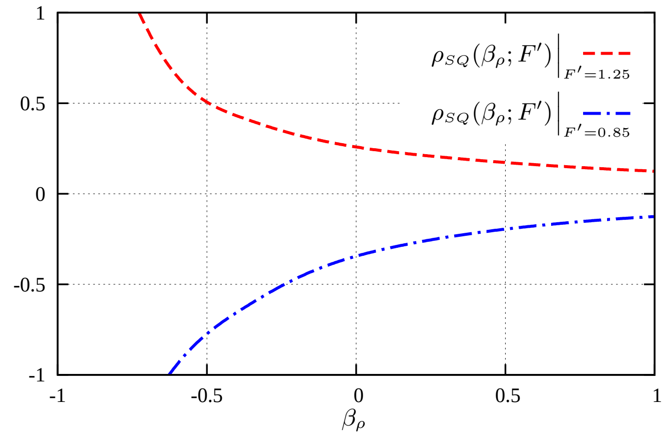

Figure 19: $\rho_{SQ}(\beta_\rho; F')$ as a function of β_ρ for $F' = 1.25$ (upper line) and for $F' = 0.85$ (lower line). The apparent mirror symmetry about the horizontal axis is pure coincidence.

for $F' = 1.25$, solutions for ρ_{SQ} only exist for $\beta_\rho \gtrsim -0.72$, and for $F' = 0.85$, only for $\beta_\rho \gtrsim -0.62$. Since we are interested in the effect of varying β_ρ on quanto option prices for any one given quanto forward $\tilde{F} := F'$, we choose the two scenarios ($\beta_\rho = -0.5, \rho_{SQ} = -0.774$) and ($\beta_\rho = 1, \rho_{SQ} = -0.126$) for $F' = 0.85$, and show the results in figure 20. Also, we choose (β_ρ=$-0.5, \rho_{SQ}$=0.506) and (β_ρ=$1, \rho_{SQ}$=0.126) for $F' = 1.25$, shown in figure 21. This gives us two examples with (almost) the most extremely possible values for β_ρ for $\tilde{F} = F' < F$, and two for $\tilde{F} = F' > F$.

To highlight the diversity of the correlation structure throughout these examples, we mention explicitly the two different sets of correlation coefficients for figure 20, namely

$\beta_\rho = -0.5$, $\rho_{SQ} = -0.774$, $F' = 0.85$:

$\rho_{\cdot\cdot}$	ξ_S	y_S	ξ_Q	y_Q
ξ_S	1	0.25	−0.774	0
y_S	0.25	1	−0.193	−0.484
ξ_Q	−0.774	−0.193	1	0
y_Q	0	−0.484	0	1

and

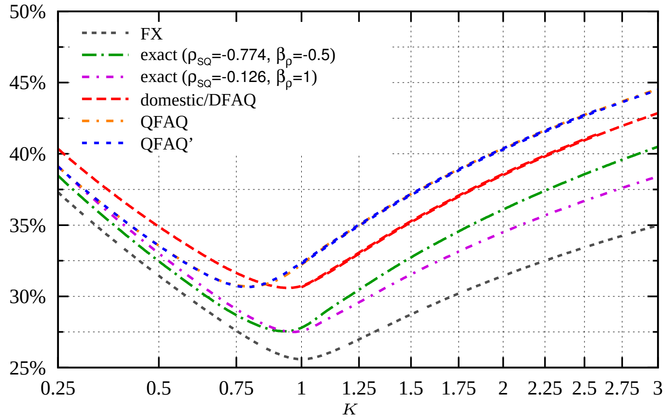

Figure 20: Implied volatilities for parameter set 5 matching $\tilde{F} = F'$ for given $F' = 0.85$.

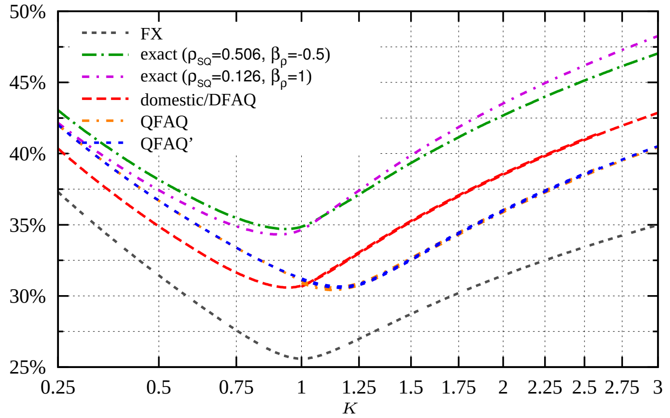

Figure 21: Implied volatilities for parameter set 5 matching $\tilde{F} = F'$ for given $F' = 1.25$.

$\beta_\rho = 1$, $\rho_{SQ} = -0.126$, $F' = 0.85$:

$\rho_{\cdot\cdot}$	ξ_S	y_S	ξ_Q	y_Q
ξ_S	1	0.25	−0.126	0
y_S	0.25	1	−0.032	0.9682
ξ_Q	−0.126	−0.032	1	0
y_Q	0	0.9682	0	1

,

and for figure 21 we have

$\beta_\rho = -0.5$, $\rho_{SQ} = 0.506$, $F' = 1.25$:

$\rho_{\cdot\cdot}$	ξ_S	y_S	ξ_Q	y_Q
ξ_S	1	0.25	0.506	0
y_S	0.25	1	0.127	−0.484
ξ_Q	0.506	0.127	1	0
y_Q	0	−0.484	0	1

and

$\beta_\rho = 1$, $\rho_{SQ} = 0.126$, $F' = 1.25$:

$\rho_{\cdot\cdot}$	ξ_S	y_S	ξ_Q	y_Q
ξ_S	1	0.25	0.126	0
y_S	0.25	1	0.032	0.9682
ξ_Q	0.126	0.032	1	0
y_Q	0	0.9682	0	1

.

We make the following observations:-

1. As expected, in both diagrams, the lines for *DFAQ* are in the exact same location, both for out-of-the-money calls (high strikes) and for out-of-the-money puts (low strikes). This merely indicates that the calibration of $\tilde{F}$ to the respectively given target value F' succeeded. Since F' differs between the first and the second example, this does of course not mean that actual option prices associated with *DFAQ* are the same across both diagrams.

2. Since the *DFAQ* line is by calibration the same as the *domestic* line, they are marked as one and the same in the graphs.

3. *QFAQ* and *QFAQ'* agree well throughout. This merely indicates that we have no numerical discrepancy between the methodology of adjusting the spot to the forward in a drift-free setting, or adjusting the drift to match the forward, since, as mentioned before, when the local volatility coefficient β_S is equal to 1, the two approaches are mathematically identical.

4. Recall that all of the curves denoted as *exact*, *DFAQ*, and *QFAQ/QFAQ'* are implied with the same quanto forward $\tilde{F} \equiv F'$.

5. For both diagrams, *QFAQ/QFAQ'* appears to give a better approximation for out-of-the-money puts, and *DFAQ* appears to give better approximations for out-of-the-money calls.

6. Near the money, i.e., say, for $K \in [0.75, 1.25]$, the two *exact* lines are in reasonably good agreement with each other in figure 20, and, likewise, in figure 21. Even for strikes very far away from the money, the *exact* implied volatility values associated with $\beta_\rho = -0.5$ and $\beta_\rho = 1$ are in reasonable agreement given that we have chosen an overall rather extreme set of parameters otherwise.

7. The accuracy of all tested ad-hoc approximations appears to be poor for options near the money and even poorer for out-of-the-money call options. We note that this is to be seen in the context of the fact that parameter set 5 constitutes an extreme test.

8. For far out-of-the money put options, the *QFAQ/QFAQ'* approach is (arguably) acceptable.

We point out that whilst we consider the above observations insightful, we have no evidence that their validity is universally applicable, which is particularly important to remember for the above points 5, 7, and 8. Having said that, we do believe that there is value for the practitioner to consider these observations as a first rule of thumb.

5 Conclusion

We have presented a numerical study of the accuracy of commonly used ad-hoc quanto option approximations in comparison in the framework of a fully fledged stochastic volatility model for both the asset and the FX process. We put particular emphasis on highlighting the wide range of quanto option prices that are attainable when only the asset's correlation with its own stochastic volatility, the FX process correlation with its own stochastic volatility, and the instantaneous asset-FX correlation are known. For the remaining freedom in the correlation matrix, we used the correlation matrix completion parametrisation suggested in [JK09] since it provides an easy means of varying all volatility-volatility correlation coefficients whilst ensuring that the full correlation matrix remains positive semi-definite. We found that for short maturities, commonly used ad-hoc approximation methods work well, but for medium to longer dated maturities, or for high volatilities, significant price discrepancies in comparison with an accurate numerical solution can be incurred. For extreme examples such as one might observe in highly volatile markets for long maturities, we noticed that the residual price uncertainty when only ρ_S, ρ_Q, and ρ_{SQ} are known can be material. We also observed that it is, in general, not possible to predict reliably which one of the investigated ad-hoc quanto adjustment conventions is going to perform best in any particular situation. We did note, though, that, overall, it is possible to say that when the asset's correlation ρ_S with its own stochastic volatility is negative, then, arguably, the *DFAQ* method performs slightly better overall, and better for out-of-the-money calls than for puts, and when $\rho_S > 0$, then, arguably, the *QFAQ/QFAQ'* methods perform better than the *DFAQ* approach, and better for out-of-the-money puts than for calls. We hasten to add, however, that these rules of thumb are speculative and reiterate the importance of the caveats given at the end of section 4.

In a further study, we considered that the actual quanto forward $\tilde{F} \equiv F'$ instead of the instantaneous process parameter ρ_{SQ} may be given. Under these circumstances, when calibrating ρ_{SQ} to the given asset and FX volatility smile as well as to the given quanto forward, we found that quanto option prices are surprisingly stable across the full attainable spectrum for the volatility-volatility control parameter β_ρ, and that even for otherwise extreme parameter settings, e.g., high volatility and long maturity. This suggests that, when quanto forward contracts are available in sufficient liquidity, hedging quanto (vanilla) options can be expected to be comparatively noise-free and insensitive to the precise choice of the remaining process parameters in a four-dimensional stochastic volatility model, and this comprises our main result. We conclude with the observation that, whilst we did not conduct tests across different stochastic volatility models, restricting the study to the double *HypHyp* model [JK07, JK08], we expect these results to be similar for other stochastic volatility models.

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