# Semi-analytic valuation of credit linked swaps in a Black-Karasinski framework

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# I. Credit linked swaps

We consider option payoffs at default of the type

$$\mathbf{1}_{\{\tau \leq T\}} \left(\phi \cdot \left(S_{\tau T}(\tau) - K\right)\right)_{+} A_{\tau T}(\tau) \ , \tag{1}$$

where:-

| | |
|---|---|
| $t$ | the valuation time |
| $\tau$ | the time of a credit event in the reference credit |
| $T$ | the maturity of an underlying swap |
| $S_{\tau T}(u)$ | the $\tau$-into-$(T-\tau)$ (forward) swap rate observed at time $u$ |
| $K$ | the swap strike |
| $A_{\tau T}(u)$ | the $\tau$ into $(T-\tau)$ annuity observed at time $u$ |
| $\phi$ | $\pm 1$ for payer/receiver swaps |

We make no explicit assumptions as to the nature of the underlying swap. It can be a conventional interest rate swap, but also a cross-currency FX swap, *etc*.

# II. Generic valuation considerations

The net present value of the credit linked swap at time $t$ is

$$V(t) = \int_t^T P_u(t) \mathsf{E}_t^{\mathcal{M}(P_u(\cdot))} \left[ \delta(u-\tau) \left( \phi \cdot (S_{uT}(u) - K) \right)_+ A_{uT}(u) \right] \mathrm{d}u \,, \tag{2}$$

$$= \int_t^T P_u(t) \mathsf{E}_t^{\mathcal{M}(P_u(\cdot))} \left[ \lambda(u) \mathrm{e}^{-\int_t^u \lambda(s)\,\mathrm{d}s} \left( \phi \cdot (S_{uT}(u) - K) \right)_+ A_{uT}(u) \right] \mathrm{d}u \tag{3}$$

$$= \int_t^T A_{uT}(t) \mathsf{E}_t^{\mathcal{M}(A_{uT}(\cdot))} \left[ \lambda(u) \mathrm{e}^{-\int_t^u \lambda(s)\,\mathrm{d}s} \left( \phi \cdot (S_{uT}(u) - K) \right)_+ \right] \mathrm{d}u \tag{4}$$

where:-

| | |
|---|---|
| $P_u(t)$ | the value of the risk-free zero coupon bond maturing at time $u$ as seen at time $t$ |
| $\lambda(u)$ | the stochastic hazard rate, *i.e.* the stochastic instantaneous default rate |
| $\mathsf{E}_t^{\mathcal{M}(N(\cdot))}[f]$ | expectation of $f$ under the measure induced by the numéraire $N(\cdot)$ in filtration $\mathcal{F}_t$ |

# III. Positive hazard rates and swaption skew control

In order to avoid the fundamental problems arising from *reverse* defaults implied by negative hazard rates, we choose to model the stochastic hazard rate as an exponential Ornstein-Uhlenbeck process

$$\lambda(u, x(u)) = \hat{\lambda}(u) \cdot \mathrm{e}^{-\frac{1}{2}\mathsf{V}_{tu}[x]+x(u)} \tag{5}$$

$$\mathrm{d}x(u) = -\varkappa_x x(u)\,\mathrm{d}u + \sigma_x(u)\,\mathrm{d}W_x(u) \tag{6}$$

similar to the well-known Black-Karasinski model for interest rates.

Note that the usual issues associated with instantaneously-lognormal short interest rate processes are of no concern for the hazard rate since expectations such as

$$\mathsf{E}_t^{\mathcal{M}(\cdot)}\left[\mathrm{e}^{\int_t^T \lambda(s,x(s))\,\mathrm{d}s}\right]$$

required for the valuation of interest rate futures contracts play no role in the valuation of credit linked swaps[1].

[1]The expectation of rollover returns is divergent when short rates evolve lognormally [HW93, SS97].

To control the model's implied swaption skew, we use a displaced exponential Ornstein-Uhlenbeck process for the swap rate in its own natural measure:-

$$S_{uT}(u, y(u)) \;=\; \hat{S}_{uT} \cdot \left[ \mathrm{e}^{-\frac{1}{2}\beta(u)^2 \mathsf{V}_{tu}[y] + \beta(u) y(u)} + \beta(u) - 1 \right] \Big/ \beta(u) \qquad (7)$$

$$\mathrm{d}y(u) \;=\; -\varkappa_y y(u)\,\mathrm{d}u + \sigma_y(u)\,\mathrm{d}W_y(u) \qquad (8)$$

- For $\beta(T) = 1$, the swaption skew for expiry $T$ is virtually flat, whereas for $\beta(T) = 0$ it resembles that generated by a Hull-White model.

- The driving processes $\mathrm{d}W_x$ and $\mathrm{d}W_y$ are correlated:

$$\mathsf{E}[\mathrm{d}W_x(u) \cdot \mathrm{d}W_y(u)] \;=\; \rho\,\mathrm{d}u \qquad (9)$$

- The deterministic expressions $\mathsf{V}_{tu}[x]$ and $\mathsf{V}_{tu}[y]$ represent the variance of the respective process variables for observation time $u$ out of filtration $\mathcal{F}_t$ with $u > t$, *i.e.*:

$$\mathsf{V}_{tu}[\aleph] \;:=\; \int_t^u \mathrm{e}^{-2\varkappa_\aleph (u-s)} \sigma_\aleph(s)^2\,\mathrm{d}s \qquad \text{for} \qquad \aleph = x, y \qquad (10)$$

# IV. Forward and swap measure calibration

A convenient way of calibrating the credit part of this model is to match a given term structure of $T_i$-forward measured survival probabilities

$$G_{T_i}(t) \;:=\; \frac{\tilde{P}_{T_i}(t)}{P_{T_i}(t)} \tag{11}$$

with $\tilde{P}_{T_i}(t)$ denoting the risky zero coupon bond maturing at time $T_i$ conditional on survival until $t$.

The function $G_{T_i}(t)$ can be specified parametrically through

$$G_{T_i}(t) := \mathrm{e}^{-\int_t^{T_i} \bar{\lambda}(s)\,\mathrm{d}s} \tag{12}$$

where $\bar{\lambda}(T_i)$ is the term structure of implied default rates supplied to represent $T_i$-forward measured survival probabilities.

The calibration procedure entails a numerical search for a suitable functional fit of the amplitude term structure $\hat{\lambda}(\cdot)$ such that

$$P_{T_i}(t) \cdot G_{T_i}(t) \;=\; N(t) \cdot \mathsf{E}_t^{\mathcal{M}(N(\cdot))}\left[ \mathrm{e}^{-\int_t^{T_i} \lambda(s,x(s))\,\mathrm{d}s} \cdot \frac{1}{N(T_i)} \right] \;. \tag{13}$$

for the chosen numéraire $N(\cdot)$. Note that this means that the calibrated function $\hat{\lambda}(\cdot)$ is a functional of the chosen numéraire.

**Measure independent calibration**

When hazard rates and swap rates are independent, or when the annuity does not depend on neither swap nor hazard rates, the amplitude term structure, $\hat{\lambda}(\cdot)$ in (5), can be calibrated by bootstrapping the constraints

$$G_{T_i}(t) \;= \mathsf{E}_t^{\mathcal{M}(P_{T_i}(\cdot))}\left[ \mathrm{e}^{-\int_t^{T_i} \lambda(s,x(s))\,\mathrm{d}s} \right] \tag{14}$$

$$= \mathsf{E}_t^{\mathcal{M}(A_{T_i T}(\cdot))}\left[ \mathrm{e}^{-\int_t^{T_i} \lambda(s,x(s))\,\mathrm{d}s} \right] \;. \tag{15}$$

## Interest rate annuity measure calibration

In general, we also allow for the hazard rate and the underlying swap rate to be correlated. When the underlying swap rate is an interest swap rate, the annuity function $A_{uT}(\cdot)$ is inevitably sensitive to changes in the swap rate $S_{uT}(\cdot)$. In that case, we can calibrate $\hat{\lambda}(\cdot)$ by matching

$$G_{t_i}(t) = \frac{A_{t_iT}(t)}{P_{t_i}(t)} \mathsf{E}_t^{\mathcal{M}(A_{t_iT}(\cdot))}\left[\mathrm{e}^{-\int_t^{t_i}\lambda(s,x(s))\,\mathrm{d}s} \Big/ A_{t_iT}(t_i)\right] \ . \tag{16}$$

In other words, since we ultimately aim to compute the expectation inside the integral in equation (4) in the annuity measure induced by $A_{uT}$, we calibrate $\hat{\lambda}(u)$ accordingly.

Since the value of the annuity $A_{uT}(\cdot)$ is, strictly speaking, not a deterministic function of the swap rate, we cannot do this unambiguously without further approximations or assumptions.

An approximation that is both convenient and commonly accepted as sufficiently accurate for practical purposes to relate the inverse of an annuity to its defining swap rate is a first order expansion of the constant-yield-to-maturity rule used for cash settled swaptions [Hag03], also known as *linear swap rate model* [HK00]:

$$\frac{1}{A_{uT}(u)} = \frac{P_u(u)}{A_{uT}(u)} \approx c_1 + c_2 \cdot S_{uT}(u) \tag{17}$$

with

$$c_1 = \frac{1}{T-u} \qquad\qquad c_2 = \frac{P_u(t)/\,A_{uT}(t) - c_1}{S_{uT}(t)} \ . \tag{18}$$

Applying this approximation to (16) gives us the calibration equation

$$G_{T_i}(t) \approx \mathsf{E}_t^{\mathcal{M}(A_{T_iT}(\cdot))}\left[\mathrm{e}^{-\int_t^{T_i}\lambda(s,x(s))\,\mathrm{d}s}\cdot\left(\left[1-\frac{S_{T_iT}(T_i)}{S_{T_iT}(t)}\right]\frac{A_{T_iT}(t)}{P_{T_i}(t)\cdot(T-T_i)}+\frac{S_{T_iT}(T_i)}{S_{T_iT}(t)}\right)\right] \ . \tag{19}$$

Note that for $T_i = T$, this reduces to

$$G_T(t) \;=\; \mathsf{E}_t^{\mathcal{M}(A_{TT}(\cdot))}\left[\mathrm{e}^{-\int_t^T \lambda(s,x(s))\,\mathrm{d}s}\right] \tag{20}$$

which is not surprising since $\mathcal{M}(P_T(\cdot)) \equiv \mathcal{M}(A_{TT}(\cdot))$.

The calibration equation (19) and the main pricing equation (3) are evaluated in the same fashion by conditioning on the evolution of the hazard rate $\lambda(s, x(s))$.

# V. Valuation by conditioning on one of the driving processes

To compute (3), we change the order of integration in time and expectation over all possible evolutions of the standard Wiener process that drives the hazard rate:-

$$V(t) = \int_{\mathbb{W}_x} \Gamma[x] \;\mathrm{d}\mu(x) \tag{21}$$

$$\Gamma[x] := \int_t^T A_{uT}(t)\lambda(u,x(u))\mathrm{e}^{-\int_t^u \lambda(s,x(s))\,\mathrm{d}s}\,\Omega(t,u,x(u))\,\mathrm{d}u \tag{22}$$

$$\Omega(t,u,x(u)) := \mathsf{E}_t^{\mathcal{M}(A_{uT}(\cdot))}\left[(\phi\cdot(S_{uT}(u,y(u))-K))_+ \middle| \, x(u)\right] \tag{23}$$

For the calculation of $\Omega(t,u,x(u))$, we use the fact that $y(u)$ conditional on $x(u)$ is a Gaussian variate with known mean and variance.

Assuming, without loss of generality, that $x(t) = y(t) = 0$, we can use the Cholesky decomposition

$$y(u)|x(u) \sim \frac{\mathsf{Cov}_{tu}[x,y]}{\mathsf{V}_{tu}[x]} \cdot x(u) + \hat{\sigma}_z(u) \cdot \sqrt{u-t} \cdot z \tag{24}$$

with

$$\hat{\sigma}_z(u)^2 \cdot (u-t) = \frac{\mathsf{V}_{tu}[x]\,\mathsf{V}_{tu}[y] - \mathsf{Cov}_{tu}[x,y]^2}{\mathsf{V}_{tu}[x]} \tag{25}$$

$$\mathsf{Cov}_{tu}[x,y] = \int_t^u \mathrm{e}^{-(\varkappa_x+\varkappa_y)(u-s)} \sigma_x(s)\sigma_y(s)\rho\,\mathrm{d}s \tag{26}$$

$$z \sim \mathcal{N}(0,1) \; . \tag{27}$$

to calculate $\Omega(t, u, x(u))$ analytically:

$$\Omega(t,u,x(u)) = \int_{-\infty}^{+\infty} \left( \phi \cdot \left( \frac{\hat{S}_{uT}}{\beta(u)} \mathrm{e}^{-\frac{1}{2}\beta(u)^2 \mathsf{V}_{tu}[y] + \beta(u) y(u)} - \frac{1-\beta(u)}{\beta(u)} \hat{S}_{uT} - K \right) \right)_+ \varphi(z)\, \mathrm{d}z$$
$$= \int_{-\infty}^{+\infty} \left( \phi \cdot \left( \frac{\hat{S}'_{uT}}{\beta(u)} \mathrm{e}^{-\frac{1}{2}\beta(u)^2 \hat{\sigma}_z(u)^2 (u-t) + \beta(u)\hat{\sigma}_z(u) \cdot \sqrt{u-t} \cdot z} - K'(u) \right) \right)_+ \varphi(z)\, \mathrm{d}z \tag{28}$$

with

$$\varphi(z) := \mathrm{e}^{-\frac{1}{2}z^2} \Big/ \sqrt{2\pi} \tag{29}$$

$$K'(u) := K + \frac{1-\beta(u)}{\beta(u)} \hat{S}_{uT} \tag{30}$$

$$\hat{S}'_{uT} := \hat{S}_{uT} \mathrm{e}^{-\frac{1}{2}\beta(u)^2 \frac{\mathsf{Cov}_{tu}[x,y]^2}{\mathsf{V}_{tu}[x]} + \beta(u) \frac{\mathsf{Cov}_{tu}[x,y]}{\mathsf{V}_{tu}[x]} x(u)} \tag{31}$$

This means that $\Omega(t, u, x(u))$ is equivalent to the Black-Scholes closed form expression with the following inputs:

- strike = $K'(u)$,
- forward = $\hat{S}'_{uT} \Big/ \beta(u)$,
- volatility = $\beta(u)\hat{\sigma}_z$,
- time to expiry = $u - t$.

The computation of the $x$-path conditional integral $\Gamma[x]$ given by (22) can be done with any appropriate temporal quadrature scheme once we have generated a discretized standard Wiener process path and selected an interpolation rule in between the discretization points for $x$.

# VI. Ornstein-Uhlenbeck process path space quadrature

The integration over all possible path evolutions $x$ is the most challenging part. It can be done in various ways.

The most straightforward approach is probably a Monte Carlo simulation.

However, since, for the purpose of valuing credit linked swaps, the path conditional functional $\Gamma[x]$ turns out to be rather benign, it is also possible to devise a rapidly convergent *spectral quadrature scheme over the entire process path space*.

Since the functional $\Gamma[x]$ depends on the $x$-process path as an *integral* over time along a function of $x$, most of the value of the integral $\Gamma[x]$ is contained in the lowest frequency modes of $x$.

This holds true particularly for low mean reversion values $\varkappa_x$, or more generally, for low values of $\varkappa_x T$ ($\lesssim 1$).

An analytical approximation to the spectral decomposition of the Ornstein-Uhlenbeck process $x$ generated by (6) for constant $\varkappa_x$ and constant $\sigma_x$ from $t = 0$ to $T$ is given by

$$x(u) = \sigma_x \sum_l a_l(u) \cdot z_l \qquad z_l \sim \mathcal{N}(0,1) \tag{32}$$

$$a_l(u) = \frac{\varkappa_x \cdot \cos \omega_l u + \omega_l \cdot \sin \omega_l u - \varkappa_x \mathrm{e}^{-\varkappa_x u}}{\omega_l^2 + \varkappa_x^2} \cdot \sqrt{\frac{2}{T}} \qquad \omega_l = \frac{2l-1}{2T} \cdot \pi \tag{33}$$

where we have assumed, wlog, that $x(t) = 0$.

In general, when $\sigma_x(u)$ is not constant, a spectral decomposition of the auto-covariance matrix of the Ornstein-Uhlenbeck process $x$ over a set of discretisation points $\{t_i\}$ in time can be computed efficiently from

$$\text{Cov}[x(t_i), x(t_j)] = \int_0^{\min(t_i,t_j)} \mathrm{e}^{-\varkappa_x(t_i+t_j-2s)} \sigma_x(s)^2 \, \mathrm{d}s \tag{34}$$

using the well-known routines `tred2`, `tqli`, and `eigsrt` described in [PTVF92], or any of `dsyev`, `dsyevd`, `dspev`, or `dspevd` in Lapack [ABB+95].

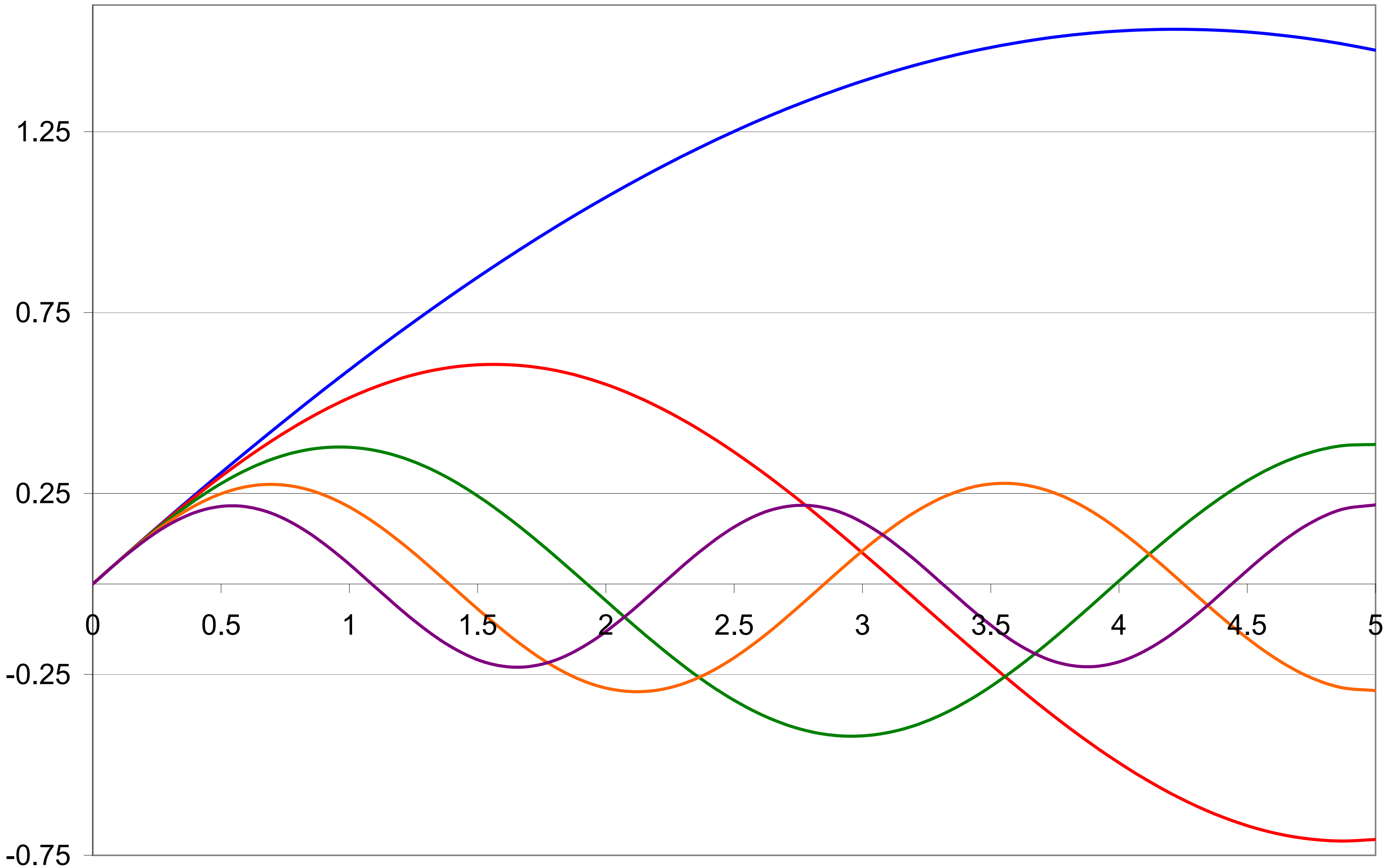


Figure VI.1. The first five Ornstein-Uhlenbeck modes for $\sigma = 50\%$, $\varkappa = 10\%$, $T = 5$.

Truncating the number of modes to $m$, we view any $x$-process path as a functional of $m$ standard normal variates $\{z_1, ..., z_m\}$ :

$$x(u) \;\rightarrow\; x(u; z_1, ..., z_m) \tag{35}$$

Denoting $w_{jn}$ as the $j$-th Gauss-Hermite weight of a one-dimensional quadrature over $n$ nodes, and $z_{jn}$ as the associated Gauss-Hermite root, the spectral appoximation to the value of the contract is given by

$$V \;=\; \sum_{j_1=1}^{n_1} w_{j_1 n_1} \sum_{j_2=1}^{n_2} w_{j_2 n_2} \cdots \sum_{j_m=1}^{n_m} w_{j_m n_m} \Gamma[x(\cdot; z_{j_1 n_1}, ..., z_{j_m n_m})] \tag{36}$$

where we have allowed for a different number of Gauss-Hermite nodes for each of the contributing modes.

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